Linear Algebra Fall 2008

Teaching, grading policies, homeworks, and so on are to be found at:

mathx.kaist.ac.kr/~schoi/teaching.html.

Check for updates each week.

Course HPs

- mathsci.kaist.ac.kr: homework, questions and answers, schedules, exam scores, grades
- mathx.kaist.ac.kr/~schoi: homework, schedules, important lecture policies(수업방침), sample exams, class assistants, how to survive…

Purpose of mathematics

- Mathematics originates with Greeks. Babylonians and Egyptians had many computations but no proofs.
- A common sense proof is a way to justify your reasoning systematically so that the truth is democratically verified. That is, most people agree with the statements. (Science starts with Ionians in the same way.)
- It is generally considered that Greeks valued reason, logic, and arguments very much for some reasons. These form foundations of the Roman age and the modern age.
- Greeks started a system of definitions, theorems, and proofs. Plato is said to be the last of the Pythagorians. This forms a beginning of the western civilizations.

We now grow out of high school mathematics

- Mathematics are taught as means to compute numbers. This is useful and very relevant to society and is what politicians and company presidents want….
- The real purpose of mathematics is to verify results and organize them into an understandable system of theorems and proofs.
- In this course, we begin. This is an opportunity for you to be more than just an engineer computing out numbers without understanding.
- A system of knowledge will aid you in finding right computations to do and verify your results often.

What is a proof?

- Proofs can be given only in a system with given axioms and logical operations as in Euclide's Geometry.
- In the system, we have a way of organizing ideas so that experts agree and are not confused about the validity (Unlike the19th century Alg. Geometers, Riemann, so on)
- A rigorous proof is a certification of validity which we can rely on.
- Experimental scientists claim to use experiments to verify their results but science always involves models and assumptions which are not verifiable by experiments only.

A rigorous proof mathematics is very different

- In other fields, such as engineering, biology, statistics, chemisty, physics, many of reasonings that occur are often not acceptable to mathematics.
- What Science does is to set up some loose system which can be verified or falsified. This uses tradition and logic and imaginations. Often precise logical steps are missing.
- In pure mathematical proofs, we strive to eliminate any gaps in logic however plausible the reasoning might be. (The proof of Fermat conjecture by Weils.)
- In applying mathematics to other fields, one needs to be careful about this distinction. Often, this is not followed completely. The other fields rely on authority more.

Learning rigorous mathematics

- The method of theorem and proof presentation of mathematics is formal and maybe is not the best way to learn.
- Communications with other people help.
- Abstract notions can be understood by specifying. That is, find examples.
- Proofs can be more easily understood by using specific examples.
- Doing exercises.

Greek mathematics

- Pythagoras: first introduced formal methods of proofs. Proved Pythagoras theorem. The existence of dodecahedron, …. Tried to show that universe is made of numbers.
- Plato: Tried to develop geometry to understand everything
- Euclides: A system of geometry
- Archimedes: integration, series, physics of lever, pulleys…
- Greek mathematics became a lost subject in the dark ages. Maybe people lost interest in thinking…

Beginning of abstract mathematics

- In 17th century, Newton discovered calculus and invented Newtonian mechanics. There were many applications. A golden age of mathematics in Europe 17-20th century follows.
- By 19th century, so much mathematics were developed and confusions began to arise. Existence questions were unanswerable in many cases.

Sets and logic

- Frege started logic and set theory. Everything should be put in logical fom.
- Russell, Whitehead tried to give foundations of mathematics using logic and set theory.
- Hilbert thought that this was possible.
- Godel found some problems with it.
- Brouwer began intuitionism. However, today most mathematicians are not following intuitionism.
- Most mathematicians ignore many subtle issues in mathematical logic

What is an abstract mathematics?

- We pretend that only sets exist and logic is the only means to study sets.
- From the set theory, we build objects such as numbers, vectors, functions so on and introduce definitions about them and study their relationships to one another.
- We prove theorems, lemmas, corollaries using logic and definitions.
- These mathematical objects and results are applied in many general situations by making concrete things abstract and conversely.

Why use abstract notions?

- An abstract notion stands for many things at the same time. Thus this reduces the amount of thinking and working. (The object oriented programming in computer science)
- Sometimes abstract objects can be viewed as another types of abstract objects. This gives us much freedom.
- Many problems can be viewed completely elementarily if interpreted differently in an abstract manner.
- If abstract ideas do work, it brings significant improvements.
- A downside is that abstract notions can be too specific to be used in situations where there are many missing knowledge and require human expertise and feelings instead of logical thinking.

Logic: the Methods of proofs

(more at old logics notes at the course HP)

- To prove P → Q: Assume P is true and then prove Q.
  - Given -, -, | Goal P → Q
  - Given -,-, P | Goal Q

    (Direct proof)

- Convert to ~Q ➔ ~P: Assume Q is false and prove P is false.
  - Given   Goal
    -,-   P ➔ Q
  - Given   Goal
    -,-, ~Q   ~P
- Example: $a, b, c \in \mathbf{R}. \quad a > b.$ Prove $ac \leq bc \rightarrow c \leq 0$
  - Given: $a, b, c \in \mathbf{R}. \quad a > b.$ Goal $ac \leq bc \rightarrow c \leq 0$
  - Given: $a, b, c \in \mathbf{R}, a > b, c > 0$ Goal: $ac > bc$

- To prove ~P: 1. Re-express in positive form. 2. Assume P and reach a contradiction.
  - Given: -,-   Goal: ~P
  - Given:-,-. P   Goal: contradiction
- To use P➔Q:
  - modus ponens: P, P➔Q : Q
  - modus tollens: P➔Q, ~Q; ~P
- To prove a goal of form P^Q: Prove P and Q separately.
- To use P^Q: Given as separate P and Q.
- To prove $P \leftrightarrow Q$: Prove P➔Q and Q➔P.
- To use $P \leftrightarrow Q$ Given as separate P➔Q and Q➔P

- To use a given of form $P \vee Q$:
  Divide into cases: 1. Assume P and case 2. Assume Q.
  - Given $P \vee Q$ Goal: ---
  - Case 1: Given P, Goal: --
  - Case 2: Given Q, Goal:---
- To prove a goal of form $P \vee Q$

  (1) Either prove P or prove Q

  (2) Assume P is false and show Q is true

- Example: $x \in \mathbf{R}$, If $x^2 \geq x$, then $x \leq 0$ or $x \geq 1$.
  - Given: $x^2 \geq x$ Goal: $x \leq 0$ or $x \geq 1$.
  - Given: $x^2 \geq x, x > 0$ Goal: $x \geq 1$

(divide by x here.)

- To prove a goal of form $\forall x P(x)$

  Given -,- Goal $\forall x P(x)$

  Given -,-, x arbitrary Goal P(x)

- To prove a goal of from $\exists x P(x)$

  Given -,- Goal $\exists x P(x)$

  Given -,-, x guessed Goal P(x)

- Example: If $x > 0$, then $\exists y, \in \mathbf{R} y(y+1) = x.$
  - Given x>0, Goal: $\exists y, \in \mathbf{R} y(y+1) = x.$
  - Guess work

$$y(y+1) = x. \quad y^2 + y - x = 0, y = (-1 \pm \sqrt{1+4x})/2.$$

  - Given $y = (-1 + \sqrt{1+4x})/2 > 0$

Goal y(y+1)=x

- To use a given of form $\exists x P(x)$
  - x_o introduce new variable
    $P(x_o)$ existential instantiation
- To use a given of form $\forall x P(x)$

We can plug in any value a for x.

Course outline

- Abstract vector spaces and linear transformations
  - Review matrices: solving equations by row operations. Reduced forms.
  - Vector spaces: abstract device
  - Linear transformations
- Classifications of linear transformations: invariants.
  - Polynomials: Ideals, generators
  - Determinants: invariant of linear maps
  - Elementary canonical forms
  - Rational form, Jordan forms
- Inner product spaces

Purpose of the course

- How to prove things: introductions to pure mathematics (and applied).
- Understand abstract notions. Using and finding examples
- Understand vector spaces, linear transformations
- Classify linear transformations independent of coordinate expressions.

Purpose of linear algebra

- People can add, subtract, multiply.
- Nonliear mathematics are hard
- Linearizations are good approximations (1st step)
- Linearizations are good design principles.
- Mathematical analysis, Quantum physics are linear (but infinite dimensional).
- Most of mathematics consist of linear or linear maps.

Linear algebra and higher level courses

- In algebra, field theory, rings, modules are generalizations of vector spaces or uses linear algebra. Number theory also uses linear algebra
- Linear maps are generalized to Lie groups, useful in many areas.
- Infinite dimensional generalizations give us mathematical analysis and quantum theory
- Geometries are studied locally by linear algebra

Chapter 1. Linear equations

- Fields
- A system of linear equations: Row equivalences, A row reduced echelon form, elementary matrices
- Invertible matrices