Contents

Figure 1: A wall paper group. p2: Four rotations of order 2 at vertices of tiled rectangles.

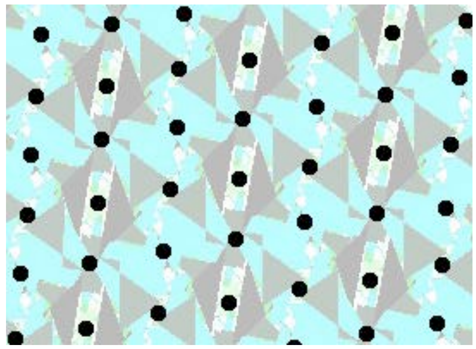
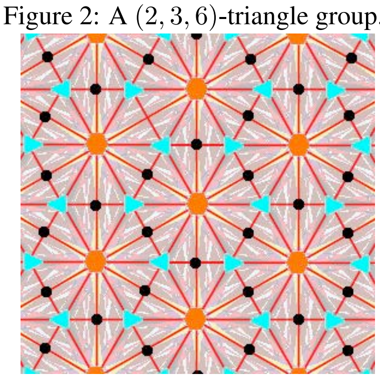


Figure 2: A $(2, 3, 6)$-triangle group.



Abstract

Abstract: In this talk, we will define the topological objects called *orbifolds*. This notion generalizes that of manifolds and is useful in some areas of mathematics related to studying discrete group actions. We give some examples and show the existence of a universal cover and define the fundamental groups. We also study 2-orbifolds by *cut and paste methods*.

1 Discrete groups

1.1 Theory of discrete groups

1.1.1 Examples

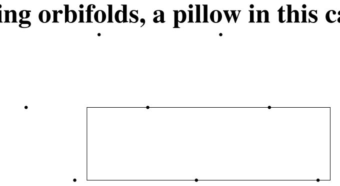
Examples

Examples

An example of constructing orbifolds, a pillow in this case.

- Consider the discrete group generated by order two rotations at (kn, lm) for $l, m \in \mathbb{Z}^2$ and fixed $k, l > 0$.
- Cut a rectangle containing two rotations on the top and the bottom sides and glue by an isometry given by the composition of the two roations.
- Then we crease the top circle and the bottom circle at the cone-points and glue. (This is called folding)
- Notice the freedom k, l and the choice of two bottom cone-points relative to the top two cone points.
- Hence, there is a 3 degrees of freedom.

An example of constructing orbifolds, a pillow in this case.



1.1.2 Definitions

Discrete groups and discrete group actions

- A *discrete group* is a group with a discrete topology. (Usually a finitely generated subgroup of a Lie group.) Any group can be made into a discrete group.
- We have many notions of a group action $\Gamma \times X \to X$:
  - The action is effective, is free
  - The action is discrete if Γ is discrete in the group of homeomorphisms of X with compact open topology.
  - The action has discrete orbits if every x has a neighborhood U so that the orbit points in U is finite.

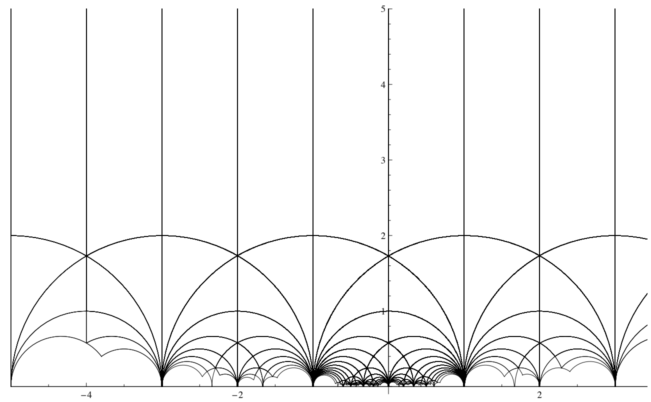
- The action is wandering if every x has a neighborhood U so that the set of elements γ of Γ so that $\gamma(U) \cap U \neq \emptyset$ is finite.
- The action is properly discontinuous if for every compact subset K the set of γ such that $K \cap \gamma(K) \neq \emptyset$ is finite.

- discrete action < discrete orbit < wandering < properly discontinuous. This is a strict relation (Assuming X is a manifold.)
- The action is wandering and free and gives manifold quotient (possibly non-Hausdorff)
- The action of Γ is free and properly discontinuous if and only if X/Γ is a manifold quotient (Hausdorff) and $X \to X/\Gamma$ is a covering map.
- Γ a discrete subgroup of a Lie group G acting on X with compact stabilizer. Then Γ acts properly discontinuously on X.
- A complete (X, G)-manifold is one isomorphic to X/Γ.
- X/Γ is isomorphic to X/Γ' iff $\Gamma' = g\Gamma g^{-1}$ for $g \in G$.
- Suppose X is simply-connected. Given a manifold M, *the set of complete (X, G)-structures on M up to (X, G)-isotopies* are in one-to-one correspondence with *the discrete representations of $\pi(M) \to G$ up to conjugations.*

Examples

- $\mathbb{R}^2 - \{O\}$ with the group generated by $g_1 : (x, y) \to (2x, y/2)$. This is a free wondering action but not properly discontinuous.
- $\mathbb{R}^2 - \{O\}$ with the group generated by $g : (x, y) \to (2x, 2y)$. (free, properly discontinuous.)
- The *modular group* $PSL(2, \mathbb{Z})$ the group of Mobius transformations or isometries of hyperbolic plane given by $z \mapsto \frac{az+b}{cz+d}$ for integer a, b, c, d and $ad - bc = 1$. `http://en.wikipedia.org/wiki/Modular_group`. This is not a free action.

The action of $PSL(2, \mathbb{Z})$.



1.2 The Poincare fundamental polyhedron theorem

1.2.1 Convex polyhedrons

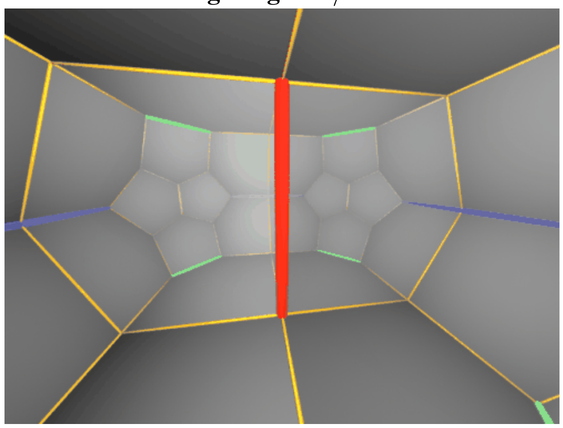
Convex polyhedrons

- A convex subset of H^n is a subset such that for any pair of points, the geodesic segment between them is in the subset.
- Using the Beltrami-Klein model, the open unit ball B, i.e., the hyperbolic space, is a subset of an affine patch $\mathbb{R}^n$. In $\mathbb{R}^n$, one can talk about convex hulls.
- Some facts about convex sets:
  - The dimension of a convex set is the least integer m such that C is contained in a unique m-plane $\hat{C}$ in H^n.
  - The interior C^o, the boundary ∂C are defined in $\hat{C}$.
  - The closure of C is in $\hat{C}$. The interior and closures are convex. They are homeomorphic to an open ball and a contractible domain of dimension equal to that of $\hat{C}$ respectively.

Examples of Convex polytopes

- A compact simplex: convex hull of $n+1$ points in H^n.
- Start from the origin expand the infinitesimal euclidean polytope from an interior point radially. That is a map sending $x_i \to \exp_O(sx_i)$ for $s > 0$ and x_i is the coordinate vertex vector of at T_0. Then take the convex hull of the image vertices. Thus for any Euclidean polytope, we obtain a one parameter family of hyperbolic polytopes.

Regular dodecahedron with all edge angles $\pi/2$



1.2.2 Fundamental domains and tessellations

Fundamental domain of discrete group action

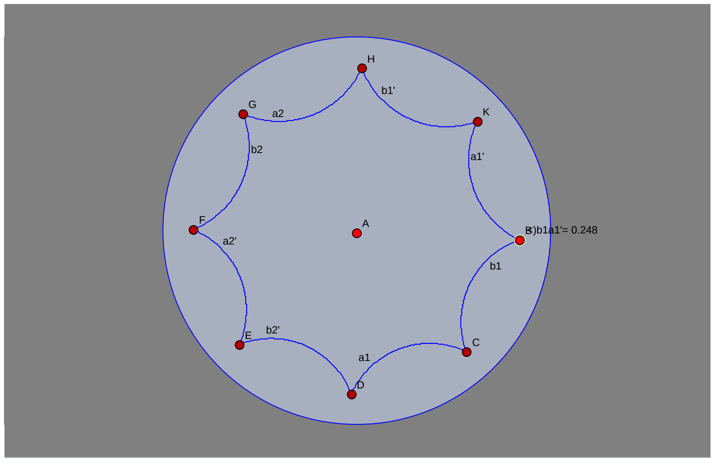
- Let Γ be a group acting on X.
- A fundamental domain for Γ is an open domain F so that $\{gF|g \in \Gamma\}$ is a collection of disjoint sets and their closures cover X.
- The fundamental domain is locally finite if the above closures are locally finite.
- The Dirichlet domain for $u \in X$ is the intersection of all $H_g(u) = \{x \in X|d(x,u) < d(x,gu)\}$. Under nice conditions, $D(u)$ is a convex fundamental polyhedron.
- The regular octahedron example of hyperbolic surface of genus 2 is an example of a Dirichlet domain with the origin as u.

Tessellations

- A *tessellation* of X is a locally-finite collection of polyhedra covering X with mutually disjoint interiors.
- Convex fundamental polyhedrons provide examples of exact tessellations.
- If P is an exact convex fundamental polyhedron of a discrete group Γ of isometries acting on X, then Γ is generated by $\Phi = \{g \in \Gamma|P \cap g(P)$ is a side of $P\}$.

1.2.3 The Poincare fundamental polyhedron theorem

The regular octahedron with side parings with vertex angle $\pi/4$.



- $(a1, D), (a1', K), (b1', K), (b1, B), (a1', B), (a1, C), (b1, C)$,
- $(b1', H), (a2, H), (a2', E), (b2', E), (b2, F), (a2', F), (a2, G)$,
- $(b2, G), (b2', D), (a1, D), (a1', K), ...$

Side pairings and Poincare fundamental polyhedron theorem

- Given a side S of an exact convex fundamental domain P, there is a unique element g_S such that $S = P \cap g_S(P)$. And $S' = g_S^{-1}(S)$ is also a side of P.
- $g_{S'} = g_S^{-1}$ since $S' = P \cap g_S^{-1}$.
- Γ-side-pairing is the set of g_S for sides S of P.
- The equivalence class at P is generated by $x \cong x'$ if there is a side-pairing sending x to x' for $x, x' \in P$.
- $[x]$ is finite and $[x] = P \cap \Gamma$.
- Cycle relations (This should be cyclic):
  - Let $S_1 = S$ for a given side S. Choose the side R of S_1. Obtain S_1'. Let S_2 be the side adjacent to S_1' so that $g_{S_1}(S_1' \cap S_2) = R$.
  - Let S_{i+1} be the side of P adjacent to S_i' such that $g_{S_i}(S_i' \cap S_{i+1}) = S_{i-1}' \cap S_i$.
- Then
  - There is an integer l such that $S_{i+l} = S_i$ for each i.
  - $\sum_{i=1}^{l} \theta(S_i', S_{i+1}) = 2\pi/k$.
  - $g_{S_1} g_{S_2} .... g_{S_l}$ has order k.

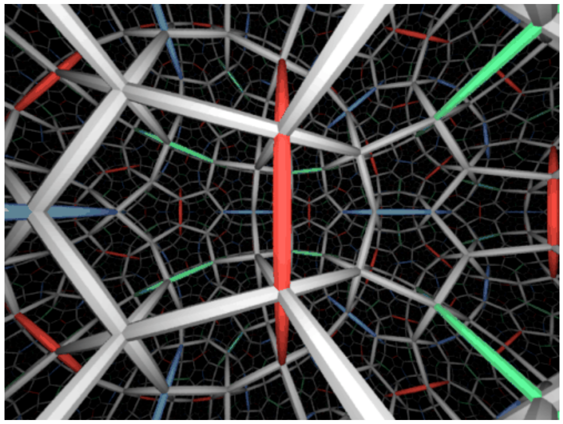
- Example: the octahedron in the hyperbolic plane giving genus 2-surface.
- The period is the number of sides coming into a given side R of codimension two.
- *Poincare fundamental polyhedron theorem* is the converse. (See Kapovich P. 80–84):
- Given a convex polyhedron P in X with side-pairing isometries satisfying the above relations, then P is the fundamental domain for the discrete group generated by the side-pairing isometries.
- If every k equals 1, then the result of the face identification is a manifold. Otherwise, we obtain orbifolds.
- The results are always complete.
- See Jeff Weeks `http://www.geometrygames.org/CurvedSpaces/index.html`

Reflection groups

- A *discrete reflection group* is a discrete subgroup in G generated by reflections in X about sides of a convex polyhedron. Then all the dihedral angles are submultiples of π.
- Then the side pairing such that each face is glued to itself by a reflection satisfies the Poincare fundamental theorem.
- The reflection group has presentation $\{S_i : (S_iS_j)^{k_{ij}}\}$ where $k_{ii} = 1$ and $k_{ij} = k_{ji}$.
- These are examples of Coxeter groups. `http://en.wikipedia.org/wiki/Coxeter_group`

Dodecahedral reflection group

One has a regular dodecahedron with all edge angles $\pi/2$ and hence it is a fundamental domain of a hyperbolic reflection group.



Triangle groups

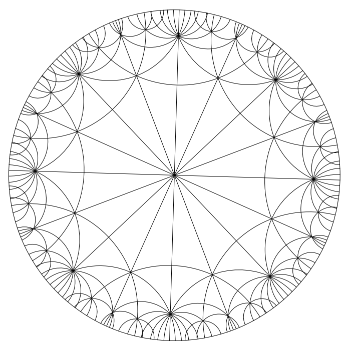
- Find a triangle in X with angles submultiples of π.
- We divide into three cases $\pi/a + \pi/b + \pi/c > \pi, = \pi, < \pi$.
- We can always find ones for any integers a, b, c.
  - $> \pi$ cases: $(2, 2, c), (2, 3, 3), (2, 3, 4), (2, 3, 5)$ corresponding to dihedral group of order $4c$, a tetrahedral group, octahedral group, and icosahedral group.
  - $= \pi$ cases: $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$.
  - $< \pi$ cases: Infinitely many hyperbolic tessellation groups.
- $(2, 4, 8)$-triangle group
- The ideal example `http://egl.math.umd.edu/software.html`

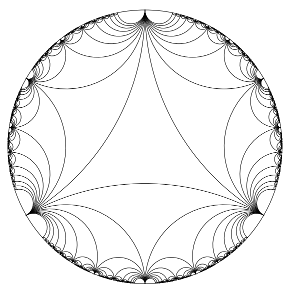
2 Pseudo-group and $\mathcal{G}$-structures

2.1 Pseudo-groups

Pseudo-groups

- In this section, we introduce pseudo-groups.
- However, we are mainly interested in classical geometries, Clifford-Klein geometries. We will be concerned with a Lie group G acting on a manifold M.
- The most obvious one is the euclidean geometry where G is the group of rigid motions acting on the euclidean space $\mathbb{R}^n$. The spherical geometry is one where G is the group $O(n+1)$ of orthogonal transformations acting on the unit sphere $\mathbf{S}^n$.





Pseudo-groups

- Topological manifolds form too large category to handle.
- To restrict the local property more, we introduce *pseudo-groups*. A pseudo-group $\mathcal{G}$ on a topological space X is the set of homeomorphisms between open sets of X so that
  - The domains of $g \in \mathcal{G}$ cover X.
  - The restriction of $g \in \mathcal{G}$ to an open subset of its domain is also in $\mathcal{G}$.
  - The composition of two elements of $\mathcal{G}$ when defined is in $\mathcal{G}$.
  - The inverse of an element of $\mathcal{G}$ is in $\mathcal{G}$.
  - If $g : U \to V$ is a homeomorphism for U, V open subsets of X. If U is a union of open sets U_α for $\alpha \in I$ for some index set I such that $g|U_\alpha$ is in $\mathcal{G}$ for each α, then g is in $\mathcal{G}$.

- The trivial pseudo-group is one where every element is a restriction of the identity $X \to X$.
- Any pseudo-group contains a trivial pseudo-group.
- The maximal pseudo-group of $\mathbb{R}^n$ is TOP, the set of all homeomorphisms between open subsets of $\mathbb{R}^n$.
- The pseudo-group C^r, $r \geq 1$, of the set of C^r-diffeomorphisms between open subsets of $\mathbb{R}^n$.
- The pseudo-group *PL* of piecewise linear homeomorphisms between open subsets of $\mathbb{R}^n$.
- (G, X)-*pseudo group*: Let G be a Lie group acting on a manifold X. Then we define the pseudo-group as the set of all pairs $(g|U, U)$ where U is the set of all open subsets of X.
- The group $\mathrm{isom}(\mathbb{R}^n)$ of rigid motions acting on $\mathbb{R}^n$ or orthogonal group $O(n+1, \mathbb{R})$ acting on $\mathbf{S}^n$ give examples.

2.2 $\mathcal{G}$-manifold

$\mathcal{G}$-manifold

A $\mathcal{G}$-*manifold* is obtained as a manifold glued with special type of gluings only in $\mathcal{G}$.

- Let $\mathcal{G}$ be a pseudo-group on $\mathbb{R}^n$. A $\mathcal{G}$-manifold is a n-manifold M with a maximal $\mathcal{G}$-atlas.

- A $\mathcal{G}$-*atlas* is a collection of charts (imbeddings) $\phi : U \to \mathbb{R}^n$ where U is an open subset of M such that whose domains cover M and any two charts are $\mathcal{G}$-compatible.
  - Two charts $(U, \phi), (V, \psi)$ are $\mathcal{G}$-*compatible* if the transition map

$$\gamma = \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \in \mathcal{G}.$$

- One can choose a locally finite $\mathcal{G}$-atlas from a given maximal one and conversely.
- A $\mathcal{G}$-*map* $f : M \to N$ for two $\mathcal{G}$-manifolds is a local homeomorphism so that if f sends a domain of a chart ϕ into a domain of a chart ψ, then

$$\psi \circ f \circ \phi^{-1} \in \mathcal{G}.$$

  That is, f is an element of $\mathcal{G}$ locally up to charts.

2.2.1 Examples

Examples

- $\mathbb{R}^n$ is a $\mathcal{G}$-manifold if $\mathcal{G}$ is a pseudo-group on $\mathbb{R}^n$.
- $f : M \to N$ be a local homeomorphism. If N has a $\mathcal{G}$-structure, then so does M so that the map is a $\mathcal{G}$-map. (A class of examples such as θ-annuli and π-annuli.)
- Let Γ be a discrete group of $\mathcal{G}$-homeomorphisms of M acting properly and freely. Then M/Γ has a $\mathcal{G}$-structure. The charts will be the charts of the lifted open sets.
- T^n has a C^r-structure and a PL-structure.
- Given (G, X) as above, a (G, X)-manifold is a $\mathcal{G}$-manifold where $\mathcal{G}$ is the restricted pseudo-group.
- A euclidean manifold is a $(\mathrm{isom}(\mathbb{R}^n), \mathbb{R}^n)$-manifold.
- A spherical manifold is a $(O(n+1), \mathbf{S}^n)$-manifold.

3 Topology of orbifolds

3.1 Definition of orbifolds

Definition of orbifolds

- Is there a good way to express the quotient spaces X/Γ?
- We should do this locally. Remember our quotient space like the pillow.
- In fact, X/Γ is given an orbifold structure.
- But orbifold structures are only slightly more general than X/Γ.

Definition of orbifolds

- X a Hausdorff second countable topological space. Let n be fixed.
- An open subset $\tilde{U}$ in $\mathbb{R}^n$ with a finite group G acting smoothly on it. A G-invariant map $\tilde{U} \to O$ for an open subset O of X inducing a homeomorphism $\tilde{U}/G \to O$. An orbifold chart is such a triple $(\tilde{U}, G, \phi)$.
- An embedding $i : (\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$ is a smooth imbedding $i : \tilde{U} \to \tilde{V}$ with $\phi = \psi \circ i$ which induces the inclusion map $U \to V$ for $U = \phi(\tilde{U})$ and $V = \phi(\tilde{V})$.
  - Equivalently, i is an imbedding inducing the inclusion map $U \to V$ and inducing an injective homomorphism $i^* : G \to H$ so that $i \circ g = i^*(g) \circ i$ for every $g \in G$. $i^*(G)$ will act on the open set that is the image of i.
  - Note here i can be changed to $h \circ i$ for any $h \in H$. The images of $h \circ i$ will be disjoint for representatives h for $H/i^*(G)$.

Definitions

- Two charts $(\tilde{U}, \phi)$ and $(\tilde{V}, \psi)$ are *compatible* if for every $x \in U \cap V$, there is an open neighborhood W of x in $U \cap V$ and a chart $(\tilde{W}, K, \mu)$ such that there are embeddings to $(\tilde{U}, \phi)$ and $(\tilde{V}, \psi)$. (One can assume W is a component of $U \cap V$.)
- If we allow $\tilde{U}$ to be an open subset of the closed upper half space, then the orbifold has boundary.

Definition of orbifold

- Since G acts smoothly, G acts freely on an open dense subset of $\tilde{U}$.
- An *orbifold atlas* on X is a family of compatible charts $\{(\tilde{U}, \phi)\}$ covering X.
- Two orbifold atlases are *compatible* if charts in one atlas are compatible with charts in the other atlas.
- Atlases form a partially ordered set. It has a maximal element.
- Given an atlas, there is a unique maximal atlas containing it.
- An orbifold is X with a maximal orbifold atlas.
- One can obtain an atlas of linear charts only: that is, charts where $\tilde{U}$ is $\mathbb{R}^n$ and $G \subset O(n)$.

3.1.1 Definitions

Definitions

- If we have $\tilde{U}$ with G acting freely, we can drop this from the atlas and replace with many charts with trivial group.
- A map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is smooth if for each point $x \in X$, there is a chart $(\tilde{U}, G, \phi)$ with $x \in U$ and a chart $(\tilde{V}, H, \psi)$ with $f(x) \in V$ so that $f(V) \subset U$ and f lifts to $\tilde{f} : \tilde{U} \to \tilde{V}$ as a smooth map.
- Two orbifolds are diffeomorphic if there is a smooth orbifold-map with a smooth inverse orbifold-map.
- $x \in X$. A local group G_x of x is obtained by taking a chart $(\tilde{U}, G, \phi)$ around x and finding the stabilizer G_y of y for an inverse image point y of x.

Definitions

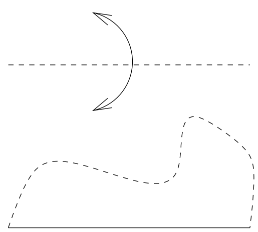
- A singular set is a set of points where G_x is not trivial.
- The subset of the singular set where G_x is constant is a relatively closed submanifold.
- Thus X becomes a stratified smooth topological space where the strata is given by the conjugacy classes of G_x.
- A suborbifold Y of an orbifold X is an imbedded subset such that for each point y in Y and and a chart $(\tilde{V}, G, \phi)$ of X for a neighborhood V of y there is a chart for y given by $(P, G|P, \phi)$ where P is a closed submanifold of $\tilde{V}$ where G acts on and $G|P$ is the image of the restriction homomorphism of G to P. (Compare with P. 35 of Adem.)

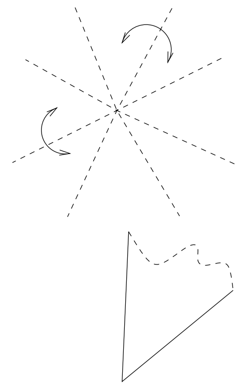
2-orbifolds

- Recall that 2-orbifold have three types of singularities: silvered points in open arcs, isolated cone-points, and isolated corner-reflector points. The singular points of a two-dimensional orbifold fall into three types:
  - (i) *A mirror point*: $\mathbb{R}^2/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by reflections on the y-axis.
  - (ii) *A cone-point of order* n: $\mathbb{R}^2/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acting by rotations by angles $2\pi m/n$ for integers m.
  - (iii) *A corner-reflector of order* n: $\mathbb{R}^2/D_n$ where D_n is the dihedral group generated by reflections about two lines meeting at an angle π/n.

2-orbifolds

- The actions here are isometries on $\mathbb{R}^2$.





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3-orbifolds

- For 3-orbifolds, the singularities are either a mirror point on 2-dimensional surface of silvered points or
- a singularity on a 1-dimensional manifolds of singularities of order n
- a singularity on the end of three 1-dimensional manifolds of singularities p, q, r where $(p, q, r) = (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5)$.
- a singularity on the vertex where three silvered surfaces meet with edges being three 1-dimensional manifolds of singularities $2p, 2q, 2r$ where $(p, q, r) = (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5)$.
- an isolated singularity of order 2.
- See Hatcher's Bianchi orbifolds.

3.1.2 Examples

Examples

- Clearly, manifolds are orbifolds.
- Let G be a finite group acting on a manifold M smoothly. Then M/G is a topological space with an orbifold structure.
- Let $M = T^n$ and $\mathbb{Z}_2$ act on it with generator acting by $-I$. For $n = 2$, $M/\mathbb{Z}_2$ is topologically a sphere and has four singular points. For $n = 4$, we obtain a Kummer surface with sixteen singular points.
- Let X be a smooth surface. Take a discrete subset. For each point, take a disk neighborhood D with a chart (D', Z_n, q) where D' is a disk and Z_n acts as a rotation with O as a fixed point and $q : D' \to D$ as a cyclic branched covering.

Examples

- Given a manifold M with boundary. We can double it as a manifold and obtain $\mathbb{Z}_2$-action. Then M has an orbifold structure.
- Take a surface and make the boundary be a union of piecewise smooth curves with corners.
  - The interior is given charts with trivial groups.
  - The interior of a boundary curve is given charts with $\mathbb{Z}_2$ as a group. (silvering)
  - The corner point is given charts with a dihedral group as a group.

Examples

- An embedded arc in the surface orbifold as above ending at two silvered boundary points is a one-dimensional suborbifold.
- Take a surface and make the boundary be a union of piecewise smooth curves with corners.
  - Some arcs are given $\mathbb{Z}_2$ as groups but not all.
  - If two such arcs meet, then the vertex is given a dihedral group as a group.
  - Then the union of the interiors of the remaining arcs is the boundary of the orbifold.
  - A nicely imbedded arc ending at a corner may not be a suborbifold unless it is in the boundary of the surface.

3.2 Differentiable structures on orbifolds

Differentiable structures on orbifolds

- Suppose we are given smooth structures on each $(\tilde{U}, G, \phi)$, i.e., $\tilde{U}$ is given a smooth structure and G is a smooth action on it. We assume that all embeddings in the atlas is smooth. Then M is given a smooth structure.
- Given a chart $(\tilde{U}, G, \phi)$, the space of smooth forms is the space of smooth forms in $\tilde{U}$ invariant under the G-action. A smooth form on the orbifold is the collection of smooth forms on each of the charts so that under embeddings they correspond.
- One can define an integral of smooth singular simplices into charts. This can be extended to any smooth simplex using partition of unity and varicentric subdivisions of the simplex.
- We can also define Riemannian structures, exponential maps, and curvatures..

Gauss-Bonnet theorem

- Assuming that X admits a finite smooth triangulation so that interior of each cell lies in singularity with locally constant isotopy groups, then we define the Euler characteristic to be
$$\chi(X) = \sum_k (-1)^{\dim s_k} 1/N_{s_k}$$
where s_k denotes the kth-cell and N_{s_k} the order of the isotropy group.
- Such a triangulation always seem to exist always. (Proved in Verona.)
- Theorem (Allendoerfer-Weil, Hopf) Let M be a compact Riemannian orbifold of even dimension m. Then
$$(2/O_m)\int_M K dw = \chi(M),$$
where O_m is the volume of the m-sphere.
- The proof essentially follows that of Chern for manifolds.

4 Covering spaces of orbifolds

4.1 Definition of the covering spaces of orbifolds

Covering spaces of orbifold

- Let X' be an orbifold with a smooth map $p : X' \to X$ so that for each point x of X, there is a connected model (U, G, ϕ) and the inverse image of $p(\psi(U))$ is a union of open sets with models isomorphic to (U, G', π) where $\pi : U \to U/G'$ is a quotient map and G' is a subgroup of G. Then $p : X' \to X$ is a *covering* and X' is a *covering orbifold* of X.
- Remember our $\mathbb{R}^2$ mapping to the pillow.

4.1.1 Examples

Examples (Thurston)

- Y a manifold. $\tilde{Y}$ a regular covering map $\tilde{p}$ with the automorphism group Γ. Let $\Gamma_i, i \in I$ be a sequence of subgroups of Γ.
  - The projection $\tilde{p}_i : \tilde{Y} \times \Gamma_i \backslash \Gamma \to \tilde{Y}$ induces a covering $p_i : (\tilde{Y} \times \Gamma_i \backslash \Gamma)/\Gamma \to \tilde{Y}/\Gamma = Y$ where Γ acts by
  $$\gamma(\tilde{x}, \Gamma_i\gamma_i) = (\gamma(\tilde{x}), \Gamma_i\gamma_i\gamma^{-1})$$
  - This is same as $\tilde{Y}/\Gamma_i \to Y$ since Γ acts transitively on both spaces.

– Fiber-products $\tilde{Y} \times \prod_{i\in I} \Gamma_i\backslash\Gamma \to \tilde{Y}$. Define left-action of Γ by

$$\gamma(\tilde{x}, (\Gamma_i\gamma_i)_{i\in I}) = (\gamma(\tilde{x}), (\Gamma_i\gamma_i\gamma^{-1})), \gamma \in \Gamma.$$

We obtain the fiber-product

$$(\tilde{Y} \times \prod_{i\in I} \Gamma_i\backslash\Gamma)/\Gamma \to \tilde{Y}/\Gamma = Y.$$

4.1.2 Definitions

Developable orbifold

- We can let Γ be a discrete group acting on *a manifold* $\tilde{Y}$ properly discontinuously but maybe not freely.
- One can find a collection X_i of coverings so that
  - $\Gamma_i = \{\gamma \in \Gamma | \gamma(X_i) = X_i\}$ is finite and if $\gamma(X_i) \cap X_i \neq \emptyset$, then γ is in Γ_i.
  - The images of X_i cover $\tilde{Y}/\Gamma$.
- $Y = \tilde{Y}/\Gamma$ has an *orbifold quotient* of $\tilde{Y}$ and Y is said to be developable.
- In the above example, we can let Γ be a discrete group acting on a manifold $\tilde{Y}$ properly discontinuously but maybe not freely. Y^f is then the fiber product of orbifold maps $\tilde{Y}/\Gamma_i \to Y$.

Cone points, corner-reflectors

- We can let D_n be a linear dihedral group acting on a disk B^2. Then B^2/D_n is given a corner-reflector orbifold structure. The maximal local group is D_n.
- One can find a collection X_i of coverings so that
  - $X_i = B^2/D_m$ where m divides n.
  - $X_i = B^2/C_m$ where m divides n where C_m is a cyclic group of order n. for a cyclic group. It has a cone-point singularity.
  - We can see these facts by paper-folding or origami.

Doubling an orbifold with mirror points

- A *mirror point* is a singular point with the stablizer group $\mathbb{Z}_2$ acting as a reflection group.
- One can double an orbifold M with mirror points so that mirror-points disappear. (The double covering orbifold is orientable.)
  - Let V_i be the neighborhoods of M with charts (U_i, G_i, ϕ_i).

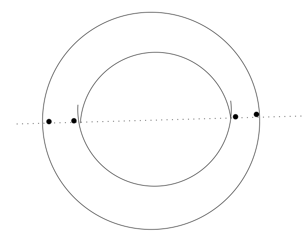
- Define new charts $(U_i \times \{-1, 1\}, G_i, \phi_i^*)$ where G_i acts by $(g(x, l) = (g(x), s(g)l)$ where $s(g)$ is 1 if g is orientation-preserving and -1 if not and ϕ_i^* is the quotient map.
- For each embedding, $i : (W, H, \psi) \to (U_i, G_i, \phi_i)$ we define a lift $(W \times \{-1, 1\}, H, \psi^*) \to (U_i \times \{-1, 1\}, G_i, \phi_i^*$. This defines the gluing.
- The result is the doubled orbifold and the local group actions are orientation preserving.
- The double covers the original orbifold with Galois group $\mathbb{Z}_2$.

- In the abstract definition, we simply let X_0' be the orientation double cover of X_0 where G-acts on X' preserving the orientation.
- For example, if we double a corner-reflector, it becomes a cone-point.

4.1.3 Examples

Some Examples

- Clearly, manifolds are orbifolds. Manifold coverings provide examples.



- The pillow covered by a torus or $\mathbb{R}^2$.
- If one double a corner-reflector, one obtains a cone-point. Thus, a disk with corner-reflectors is double-covered by a sphere with cone-points.
- The edge folded triangle covered by tori.
- Let Y be a tear-drop orbifold with a cone-point of order n. Then this cannot be covered by any other type of an orbifold and hence is a universal cover of itself.
- A sphere Y with two cone-points of order p and q which are relatively prime.
- Choose a cyclic action of Y of order m fixing the cone-point. Then Y/Z_m is an orbifold with two cone-points of order pm and qm.

4.2 Universal covering spaces

Universal covering by fiber-product

- A universal cover of an orbifold Y is an orbifold $\tilde{Y}$ covering any covering orbifold of Y.

- We will now show that the universal covering orbifold exists by using fiber-product constructions. For this we need to discuss elementary neighborhoods. An *elementary* neighborhood is an open subset with a chart components of whose inverse image are open subsets with charts.

- We can take the model open set in the chart to be simply connected.

- Then such an open set is elementary.

Fiber-product for D^n/G_i

- If V is an orbifold D^n/G for a finite group G.
  - Any covering is D^n/G_1 for a subgroup G_1 of G.
  - Given two covering orbifolds D^n/G_1 and D^n/G_2, a covering morphism is induced by $g \in G$ so that $gG_1g^{-1} \subset G_2$.
  - The covering morphism is in one-to-one correspondence with the double cosets of form G_2gG_1 for g such that $gG_1g^{-1} \subset G_2$.
  - The covering automorphism group of D^n/G' is given by $N(G_1)/G_1$.
  - Given coverings $p_i : D^n/G_i \to D^n/G$ for $G_i \subset G$ for V homeomorphic to a cell, we form a fiber-product.

$$V^f = (D^n \times \prod_{i \in I} G_i \backslash G)/G \to D^n/G$$

  - If we choose all subgroups G_i of G, then any covering of D^n/G is covered by V^f induced by projection to G_i-factor (universal property)

The construction of the fiber-product of a sequence of orbifolds

- Let $Y_i, i \in I$ be a collection of the orbifold-coverings of Y.

- We cover Y by elementary neighborhoods V_j for $j \in J$ forming a good cover.

- We take inverse images $p_i^{-1}(V_j)$ which is a disjoint union of V/G_k for some finite group G_k.

- Fix j and we form one fiber product by V/G_k by taking one from $p_i^{-1}(V_j)$ for each i.

- Fix j and we form a fiber-product of $p_i^{-1}(V_j)$, which will essentially be the disjoint union of the above fiber products indiced by the product of the component indices for each i.

- Over regular points of V_j, this is the ordinary fiber-product.

The construction of the fiber-product of a sequence of orbifolds

- Now, we wish to patch these up using imbeddings. Let $U \to V_j \cap V_k$. We can assume $U = V_j \cap V_k$ which has a convex cell as a cover.
  - We form the fiber products of $p_i^{-1}(U)$ as before which can be realized in V_j and V_k.
  - Over the regular points in V_j and V_k, they are isomorphic. Then they are isomorphic.
  - Thus, each component of the fiber-product can be identified.
- By patching, we obtain a covering Y^f of Y with the covering map p^f.

4.2.1 The construction and the properties of the universal cover

The construction of the universal cover

- The collection of cover of an orbifold is countable upto isomorphisms preserving base points. (Cover by a countable good cover and for each elementary neighborhood, there is a countable choice.)
- Take a fiber product of Y_i, $i = 1, 2, 3, ...$ The fiber-product $\tilde{Y}$ with a base point $*$. We take a connected component.
- The for any cover Y_i, there is a morphism $\tilde{Y} \to Y_i$.
- The universal cover is unique up to covering orbifold-isomorphisms by the universality property.

Properties of the universal cover

- The group of automorphisms of $\tilde{Y}$ is called the fundamental group and is denoted by $\pi_1(Y)$.
- $\pi_1(Y)$ acts transitively on $\tilde{Y}$ on fibers of $\tilde{p}^{-1}(x)$ for each x in Y. (To prove this, we choose one covering of Y from a class of base-point preserving isomorphism classes of coverings of Y. Then the universal cover with any base-point occurs will occur in the list and hence a map from $\tilde{Y}$ to it preserving base-points.)
- $\tilde{Y}/\pi_1(Y) = Y$.
- Any covering of Y is of form $\tilde{Y}/\Gamma$ for a subgroup Γ of $\pi_1(Y)$.
- The isomorphism classes of coverings of Y is the set of conjugacy classes of subgroups of $\pi_1(Y)$.
- There is a path-approach to defining the fundamental groups that we will not be considering.

Properties of the universal cover

- The group of automorphism is $N(\Gamma)/\Gamma$.
- A covering is regular if and only if Γ is normal.
- A good orbifold is an orbifold with a cover that is a manifold.
- An very good orbifold is an orbifold with a finite cover that is a manifold.
- A good orbifold has a simply-connected manifold as a universal covering space.

Induced homomorphism of the fundamental group

- Given two orbifolds Y_1 and Y_2 and an orbifold-diffeomorphism $g : Y_1 \to Y_2$. Then the lift to the universal covers $\tilde{Y}_1$ and $\tilde{Y}_2$ is also an orbifold-diffeomorphism. Furthermore, once the lift value is determined at a point, then the lift is unique.
- Also, homotopies $f_t : Y_1 \to Y_2$ of orbifold-maps lift to homotopies in the universal covering orbifolds $\tilde{f}_t : \tilde{Y}_1 \to \tilde{Y}_2$. Proof: we consider regular parts and model neighborhoods where the lift clearly exists uniquely.
- Given orbifold-diffeomorphism $f : Y \to Z$ which lift to a diffeomorphism $\tilde{f} : \tilde{Y} \to \tilde{Z}$, we obtain $f_* : \pi_1(Y) \to \pi_1(Z)$.
- If g is homotopic to f, then $g_* = f_*$.

4.2.2 Examples

Examples

- An annulus with one boundary component silvered has a fundamental group isomorphic to $Z \times Z_2$.

  The fundamental group can be computed by removing open-ball neighborhoods of the cone-points and using Van-Kampen theorem.
- Suppose that a two-dimensional orbifold has boundary and silvered points. Then remove open-ball neighborhoods of the cone-points and corner-reflector points. Then the fundamental group of remaining part can be computed by Van-Kampen theorem by taking open neighborhoods of silvered boundary arcs. Finally, adding the open-ball neighborhoods, we compute the fundamental group.
- The fundamental group of a three-dimensional orbifold can be computed similarly.

5 2-orbifolds

5.1 Classifications of 1-dimensional suborbifolds of 2-orbifolds

The triangulations of 2-orbifolds and classification

- Theorem: Any 2-orbifold is obtained from a smooth surface with corner by silvering some arcs and putting cone-points and corner-reflectors.
- A 2-orbifold is classified by the underlying smooth topology of the surface with corner and the number and orders of cone-points, corner-reflectors, and the boundary pattern of silvered arcs.
- proof: basically, strata-preserving isotopies.

Classifications of 1-dimensional suborbifolds of 2-orbifolds

- A *suborbifold* Q' on a subspace $X_{Q'} \subset X_Q$ is the subspace so that each point of $X_{Q'}$ has a neighborhood in X_Q modeled on an open subset U of $\mathbb{R}^n$ with a finite group Γ preserving $U \cap \mathbb{R}^d$ where $\mathbb{R}^d \subset \mathbb{R}^n$ is a proper subspace, so that $(U \cap \mathbb{R}^d, \Gamma')$ is in the orbifold structure of Q'.
- Here Γ' denotes the restricted group of Γ to $U \cap \mathbb{R}^d$, which is in general a quotient group.

Classifications of 1-dimensional suborbifolds of 2-orbifolds

- A compact 1-orbifold is either a closed arc, a segment, a segment with one silvered endpoint, or a segment with two silvered end-point.
- Properly and nicely imbedded 1-orbifolds in a 2-orbifold with boundary. (nice means that only boundary goes to boundary.)
  - *No silvered-point case*: An imbedded closed arc avoiding boundary or a segment with two endpoints in the boundary
  - *One silvered-point case*: A segment with silvered endpoint at a cone-point of order two or a silvered arc and the other endpoint in the boundary.
  - *Two silvered-point case*: A segment with silvered endpoints at cone-points or order two or in silvered arcs.

5.2 Orbifold Euler-characteristic for 2-orbifolds

Orbifold Euler-characteristic for 2-orbifolds due to Satake

- We define the *Euler characteristic* to be

$$\chi(X) = \sum_{c_i} (-1)^{\dim(c_i)} (1/|\Gamma(c_i)|),$$

  where c_i ranges over the open cells and $|\Gamma(c_i)|$ is the order of the group Γ_i associated with c_i.

- If X is finitely covered by another orbifold X', then $\chi(X') = r\chi(X)$ where r is the number of sheets for regular points. This follows since the sum of the order of local groups in the inverse image of the elementary neighborhood is always r.
- The Euler-characteristic of 1-orbifold: a circle O, a segment 1, a segment with one silvered-point $1/2$, a full 1-orbifold O.

Orbifold Euler-characteristic for 2-orbifolds due to Satake

- For 2-orbifolds Σ_1, Σ_2 meeting in a compact 1-orbifold Y in the interior forming a 2-orbifold Σ as a union, we have the following additivity formula:

$$\chi(\Sigma) = \chi(\Sigma_1) + \chi(\Sigma_2) - \chi(Y), \tag{1}$$

- To be verified by counting cells with weights since the orders of singular points in the boundary orbifold equal the ambient orders.

Orbifold Euler-characteristic for 2-orbifolds due to Satake

- Suppose that a 2-orbifold Σ with or without boundary has the underlying space X_Σ and m cone-points of order q_i and n corner-reflectors of order r_j and n_Σ boundary full 1-orbifolds.
- Then the following *generalized Riemann-Hurwitz formula* is very useful:

$$\chi(\Sigma) = \chi(X_\Sigma) - \sum_{i=1}^{m}\left(1 - \frac{1}{q_i}\right) - \frac{1}{2}\sum_{j=1}^{n}\left(1 - \frac{1}{r_j}\right) - \frac{1}{2}n_\Sigma, \tag{2}$$

  which is proved by a doubling argument and cutting and pasting.
- If Σ has nonempty boundary, then it is easy to show that the boundary consists of circles and full 1-orbifolds, which are mutually disjoint suborbifolds.

5.3 Definition of Splitting and sewing 2-orbifolds

Definition of Splitting and sewing 2-orbifolds

- A compact 2-orbifold is *good* except for the sphere or with one or two cone-singularities of order p and q where $p \neq q$ and a disk with one or two corner singularities of order $p, q, p \neq q$.
- The compact good 2-orbifolds are always *very good.* If $\chi \leq 0$, then it is very good.
- Let S be a very good orbifold so that its underlying space X_S is a pre-compact open surface with a path-metric admitting a compactification to a surface with boundary.

- Let $\hat{S}$ be a very good cover, that is, a finite regular cover, of S, so that S is orbifold-diffeomorphic to $\hat{S}/F$ where F is a finite group acting on $\hat{S}$.

- Since $X_{\hat{S}} = \hat{S}$ is also pre-compact and has a path-metric, complete it to obtain a compact surface $X'_{\hat{S}}$.

- $X'_{\hat{S}}/F$ with the quotient orbifold structure is said to be the *orbifold-completion* of S.

- Let S be a 2-orbifold with an embedded circle or a full 1-orbifold l in the interior of S. The completion S' of $S - l$ is said to be obtained from splitting S along l. Since $S - l$ has an embedded copy in S', we see that there exists a map $S' \to S$ sending the copy to $S - l$. Let l' denote the boundary component of S corresponding to l under the map.

- Conversely, S is said to be obtained from sewing S' *along* l'.

- If the interior of the underlying space of l lies in the interior of the underlying space of S, then the components of S' are said to be *decomposed components of* S *along* l, and we also say that S *decomposes* into S' along l.

- Of course, if l is a union of disjoint embedded circles or full 1-orbifolds, the same definition holds.

- Here we can consider S as a some open suborbifold of a 1-orbifold in consideration.

Silvering and clarifying

- There are two distinguished classes of splitting and sewing operations:

- A simple closed curve boundary component can be made into a set of mirror points and conversely in a unique manner.

- a boundary point has a neighborhood which is realized as a quotient of an open ball by a $\mathbb{Z}_2$-action generated by a reflection about a line.

- A boundary full 1-orbifold can be made into a 1-orbifold of mirror points and two corner-reflectors of order two and conversely in a unique manner: (a boundary point has a neighborhood which is a quotient space of a dihedral group of order four acting on the open ball generated by two reflections.)

- The forward process is called silvering and the reverse process clarifying.

5.4 Regular neighborhoods of 1-orbifolds

The classification of the Euler-characteristic zero orbifolds

- Let A be a compact annulus with boundary. The quotient orbifold of an annulus has Euler characteristic zero.

- From Riemann-Hurwitz equation, all of the Euler characteristic zero 2-orbifolds with nonempty boundary:

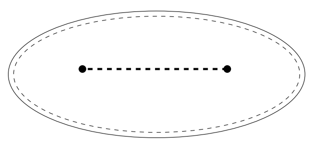
  (1) an annulus, (2) a Möbius band, (3) an annulus with one boundary component silvered (a *silvered annulus*),

  (4) a disk with two cone-points of order two with no mirror points (a $(;2,2)$-disk),

  (5) a disk with two boundary 1-orbifolds, two edges (a silvered strip),

  (6) a disk with one cone-point and one boundary full 1-orbifold (a bigon with a cone-point of order two), that is, it has only one edge, and

  (7) a disk with two corner-reflectors of order two and one boundary full 1-orbifold (a half-square). (It has three edges.)



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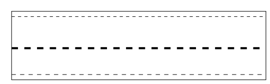

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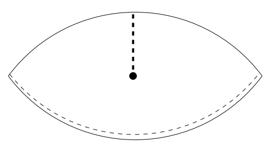

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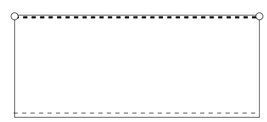

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Regular neighborhoods of 1-orbifold

- A circle or a 1-orbifold l in the interior of a 2-orbifold S, not homotopic to a point.
- l has a neighborhood of zero Euler characteristic considering its good cover.
- Since the inverse image of l consists of closed curves which represent generators.
  - For (1) and (2), l is the closed curve representing the generator of the fundamental group;
  - For (3), l is the mirror set that is a boundary component;
  - For (4), l is the arc connecting the two cone-points unique up to homotopy;
  - For (5), l is an arc connecting two interior points of two edges respectively;
  - For (6), l is an arc connecting an interior point of an edge and the cone-point of order two;
  - For (7), the edge in the topological boundary connecting the two corner-reflectors of order two.

Regular neighborhoods of 1-orbifold

- Given a 1-orbifold l and a neighborhood N of it in some ambient 2-orbifold, N is said to be a regular neighborhood if the pair (N, l) is diffeomorphic to one of the above.
- A 1-orbifold in a good 2-orbifold has a regular neighborhood which is unique up to isotopy.

5.5 Splitting and sewing on 2-orbifolds reinterpreted

Splitting on 2-orbifolds reinterpreted

- Let l be a 1-orbifold embedded in the interior of an orbifold S.
- If one removes l from the interior of a regular neighborhood, we obtain either a union of one or two open annuli, or a union of one or two open silvered strip.
- In (2)-(4), an open annulus results. For (1), a union of two open annuli results. For (6)-(7), an open silvered strip results. For (5), we obtain a union of two open silvered strips.

- These can be easily completed to be a union of one or two compact annuli or a union of one or two silvered strips respectively.
- We can complete $S - l$ in this manner: We take a closed regular neighborhood N of l in S.
- We remove $N - l$ to obtain the above types and complete it and re-identify with $S - l$ to obtain a compactified orbifold. This process is the *splitting* of S along l.

Sewing on 2-orbifolds reinterpreted

- Conversely, we can describe *sewing*: Take an open annular 2-orbifold N which is a regular neighborhood of a 1-orbifold l.
- Suppose that l is a circle. We obtain $U = N - l$ which is a union of one or two annuli.
- Take an orbifold S' with a union l' of one (resp. two) boundary components which are circles.
- Take an open regular neighborhood of l' and remove l' to obtain V.
- U and V are the same orbifold. We identify $S' - l'$ and $N - l$ along U and V.
- This gives us an orbifold S, and it is easy to see that S is obtained from S' by sewing along l'.

Identification interpretations of splitting and sewing

- In the following we describe the topological identification process of the underlying space involved in these six types of *sewings*. The orbifold structure on the sewed orbifold should be clear.
- Let an orbifold Σ have a boundary component b. (Σ is not necessarily connected.) b is either a simple closed curve or a full 1-orbifold. We find a 2-orbifold Σ'' constructed from Σ by sewing along b or another component of Σ.

- (A) Suppose that b is diffeomorphic to a circle; that is, b is a closed curve. Let Σ' be a component of the 2-orbifold Σ with boundary component b'. Suppose that there is a diffeomorphism $f : b \to b'$. Then we obtain a bigger orbifold Σ'' glued along b and b' topologically.

  (I) The construction so that Σ'' does not create any more singular point results in an orbifold Σ'' so that

  $$\Sigma'' - (\Sigma - b \cup b')$$

  is a circle with neighborhood either diffeomorphic to an annulus or a Möbius band.

  (1) In the first case, $b \neq b'$ (pasting).

  (2) In the second case, $b = b'$ and $\langle f \rangle$ is of order two without fixed points (cross-capping).

- (II) When $b = b'$, the construction so that Σ'' does introduce more singular points to occur in an orbifold Σ'' so that

  $$\Sigma'' - (\Sigma - b)$$

  is a circle of mirror points or is a full 1-orbifold with endpoints in cone-points of order two depending on whether $f : b \to b$

  (1) is the identity map (silvering), or

  (2) is of order two and has exactly two fixed points (folding).

- (B) Consider when b is a full 1-orbifold with endpoints mirror points.

  (I) Let Σ' be a component orbifold (possibly the same as one containing b) with boundary full 1-orbifold b' with endpoints mirror points where $b \neq b'$. We obtain a bigger orbifold Σ'' by gluing b and b' by a diffeomorphism $f : b \to b'$. This does not create new singular points (pasting).

(II) Suppose that $b = b'$. Let $f : b \to b$ be the attaching map. Then

(1) if f is the identity, then b is silvered and the end points are changed into corner-reflectors of order two (silvering).

(2) If f is of order two, then Σ'' has a new cone-point of order two and has one-boundary component orbifold removed away. b corresponds to a mixed type 1-orbifold in Σ' (folding).

– It is obvious how to put the orbifold structure on Σ'' using the previous descriptions using regular neighborhoods above.

6 Lecture 2: Introduction to orbifolds II: Geometry

Abstract:

In this talk, we will define geometric structures on orbifolds and define their deformation spaces and the fact that the deformation spaces are locally homeomorphic to the G-representation spaces quotient by conjugations. We introduce the Teichmuller theory of orbifolds, i.e., the deformation spaces of hyperbolic structures on 2-orbifolds, and show that they are homeomorphic to cells.

6.1 Geometric structures on orbifolds

Definition of geometric structures on orbifolds

- Let (X, G) be a pair defining a geometry. That is, G is a Lie group acting on a manifold effectively and transitively.
- Given an orbifold M, there is at least three ways to define (X, G)-*geometric structures* on M.
  - Using atlas of charts.
  - A developing map from the universal covering space.
  - A cross-section of the flat orbifold X-bundle.

Atlas of charts approach

- Given an atlas of charts for M, for each chart (U, K, ϕ) we find an X-chart $\rho : U \to X$ and an injective homomorphism $h : K \to G$ so that ρ is an equivariant map.
- For each imbedding $i : (V, H, \psi) \to (U, K, \phi)$ where V has an X-chart $\rho' : V \to X$ and equivariant with respect to an injective homomorphism $h' : H \to G$, we have
$$\rho \circ i = g \circ \rho', h'(\cdot) = gh(i^*(\cdot))g^{-1}$$
- If we simply identify with open subsets of X, the above simplifies greatly and i is a restriction of an element of g and i^* is a conjugation by g also.

- This gives us a way to build an orbifold from pieces of X.
- A maximal such atlas of X-charts is called *an* (X, G)*-structure* on M.

Atlas of charts approach

- An (X, G)*-map* $M \to N$ is a smooth map f so that for each x and $y = f(x)$, there are charts (U, K, ϕ) and (V, H, ψ) so that f sends $\phi(U)$ into $\psi(V)$ and lifts to $\tilde{f} : U \to V$ so that $\rho' \circ \tilde{f} = g \circ \rho$ and $h'(i^*(\cdot)) = gh(\cdot)g^{-1}$.
- In otherwards, f is a restriction of an element g of G up to charts with a homomorphism $K \to H$ induced by a conjugation by an element of G.

Atlas of charts approach

- (X, G)-orbifold is always good.
- Proof:
  - Basically build a germ of local (X, G)-maps from M to X which is a principal bundle and is a manifold: For each (U, K, ϕ), we build $G(U) = G \times U/K$ and a projection $G(U) \to U$. We paste these together to find $G(M)$.
  - $G(M)$ is a manifold since K acts on $G \times U$ freely.
  - The foliation given by pasting $g_0 \times U$ is a foliation by open manifolds with the same dimension as M. Each leave of the foliation is covers M.
- If G is a subrgroup of a linear group, then M is very good by Selberg's lemma.
- Thus M is a quotient $\tilde{M}/\Gamma$ where Γ contains copies of all of the local group.

The developing maps and holonomy homomorphisms

- Let $\tilde{M}$ denote the universal cover of M with a deck transformation group π.
- Then we obtain a developing map $D : \tilde{M} \to X$ by first finding an initial chart $\rho : U \to X$ and continuing by extending maps by patches.
- One uses a nice cover of $\tilde{M}$ and extend. The map is well-defined independently of which path of charts one took to arrive at a given chart.
- To show this, we need to homotopy and consider three nice charts simultaneously and the fact that M admits a real analytic structure and the charts are real analytic and hence if they agree on an open set, then they extend each other.
- This gives an (X, G)-structure on $\tilde{M}$ as well and the cover map is an (X, G)-map.

The developing maps and holonomy homomorphisms

- Since we can change the initial chart to $k \circ \rho$ for any $k \in G$, we see that $k \circ D$ is another developing map and conversely any developing map is of such form.
- Given a deck transformation $\gamma : \tilde{M} \to \tilde{M}$, we see that $D \circ \gamma$ is a developing map also and hence equals $h(\gamma) \circ D$ for some $h(\gamma) \in G$.
- The map $h : \pi \to G$ is a homomorphism, so-called the *holonomy homomorphism*.
- The pair (D, h) is said to be the development pair.
- The development pair is determined up to an action of G given by $(D, h(\cdot)) \to (g \circ D, gh(\cdot)g^{-1})$.

The developing maps and holonomy homomorphisms

- Conversely, a developing map (D, h) gives us X-charts:
- For each open chart (U, K, ψ), we lift to a component of $p^{-1}(U)$ in $\tilde{M}$ and obtain a restriction of D to the component. This gives us X-charts.
- A different choice of components gives us the compatible charts.
- Local group actions and imbeddings satisfy the desired properties.
- Thus, a development pair completely determines the (X, G)-structure on M.

6.2 The definition of the Teichmüller space of 2-orbifolds

Definition of Teichmüller spaces of 2-orbifolds

- A hyperbolic structure on a 2-orbifold is a geometric structure modeled on H^2 with the isometry group $\mathrm{PSL}(2, \mathbb{R})$.
- The Teichmüller space $\mathcal{T}(M)$ of a 2-orbifold M is the deformation space of hyperbolic structures on the 2-orbifold.
- As before, we reinterpret the space as
  - The set of diffeomorphisms $f : M \to M'$ for M an orbifold and M' a hyperbolic 2-orbifold.
  - The equivalence relation $f : M \to M'$ and $g : M \to M$" if exists a hyperbolic isometry $h : M' \to M$" so that $h \circ f$ is isotopic to g.
  - The quotient space is same as above.
- A necessary condition for an orbifold to have a hyperbolic structure is that the orbifold euler characteristic be negative: This follows from the Gauss-Bonnet theorem. Here the negative of the hyperbolic area is the Euler characteristic times 2π.

- A closed 2-orbifold with a complex structure has a unique hyperbolic structure provided it is compact and has negative Euler characteristic.
- The deformation space of complex structures on a closed 2-orbifold is identical with the Teichmuller space as defined here by the uniformization theorem.

6.3 The geometric cutting and pasting and the deformation spaces

The geometric cutting and pasting and the deformation spaces

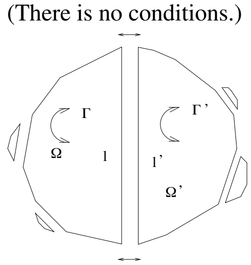
- A compact geodesic 1-orbifold without boundary points in the interior of a 2-orbifold Σ are either
  - a closed geodesic in the interior or entirely in the boundary of $|\Sigma|$ or
  - a segment with two silvered points which are either at silvered edges or cone-points of order two. The topological interior is either in the interior or entirely in the boundary of $|\Sigma|$.
- The geometric type is classified by length and the topological type. Such a geodesic 1-orbifold is covered by a closed geodesic in some cover of the 2-orbifold, which is a surface.
- The *Teichmüller space* $\mathcal{T}(I)$ for a 1-orbifold I is the product of the spaces of lengths $\mathbb{R}^+$ for components of I.
-

Geometric constructions.

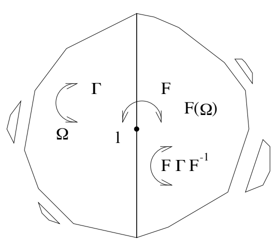
- Recall the type of topological constructions with 1-orbifolds. Suppose they are boundary components of 2-orbifolds whose components have negative Euler characteristics.
  - (A)(I) Pasting or crosscapping along simple closed curves.
  - (A)(II) Silvering or folding along a simple closed curve.
  - (B)(I) Pasting along two full 1-orbifolds.
  - (B)(II) Silvering or folding along a full 1-orbifold.
- Now we suppose that the simple closed curves and 1-orbifolds are geodesic and try to obtain geometric version of the above.

Geometric constructions.

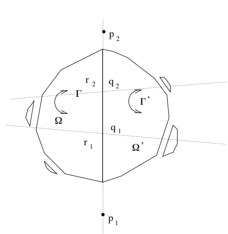
- Suppose that the involved 1-orbifolds are geodesic boundary components of a hyperbolic 2-orbifolds.

- (A)(I). For pasting two closed geodesics, we have a $\mathbb{R}$-amount of isometries to do this. They will create hyperbolic structures inequivalent in the Teichmüller space. (Here the length of two closed geodsics have to be the same.)
- (A)(I) For cross-capping, we have a unique isometry. The isometry has to be a slide reflection of distance half the length of the closed geodesic. (There is no conditions.)
- 

Geometric constructions.

- (A)(II). For folding a closed geodesics, we have a $\mathbb{R}$-amount of isometries to do this. They will create hyperbolic structures inequivalent in the Teichmüller space. The choice depends on the choice of two fixed points of the pasting map. The distance is the half of length of the closed geodesic. (no condition)
- (A)(II) For silvering, we have unique isometry to do this. (no condition)
- 

Geometric constructions.

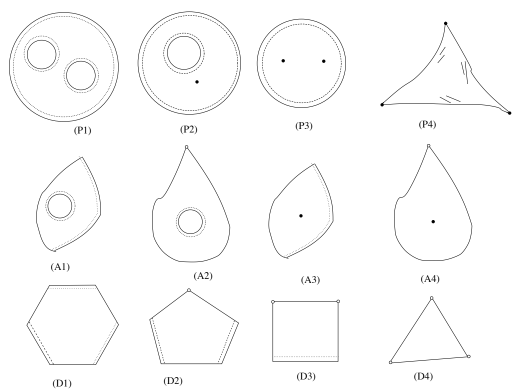
- (B)(I). For pasting along two geodesic full 1-orbifolds, We have a unique way to do this. The lengths of the orbifolds have to be the same.
- (B)(II) For silvering and folding, we have unique isometry to do this. (no condition)
- 

6.4 The decomposition of 2-orbifolds into elementary orbifolds.

Topological decomposition of hyperbolic 2-orbifolds into elementary orbifolds along geodesic 1-orbifolds.

- Suppose that Σ is a compact hyperbolic orbifold with $\chi(\Sigma) < 0$ and geodesic boundary.
- Let $c_1, \ldots, c_n$ be a mutually disjoint collection of simple closed curves or 1-orbifolds so that the orbifold Euler characteristic of the completion of each component of $\Sigma - c_1 - \cdots - c_n$ is negative.
- Then $c_1, \ldots, c_n$ are isotopic to simple closed geodesics or geodesic full 1-orbifolds $d_1, \ldots, d_n$ respectively where $d_1, \ldots, d_n$ are mutually disjoint.

The diagram for elementary orbifolds

- 

  The elementary orbifolds. Arcs with dotted arcs next to them indicate boundary components. Black points indicate cone-points and white points the corner-reflectors.

Elementary 2-orbifolds.

We require the boundary components be geodesics.

(P1) A pair-of-pants.

(P2) An annulus with one cone-point of order n. ($A(;n)$)

(P3) A disk with two cone-points of order p, q, one of which is greater than 2. ($D(;p,q)$)

(P4) A sphere with three cone-points of order p, q, r where $1/p + 1/q + 1/r < 1$. ($\mathbf{S}^2(;p,q,r)$)

(A1) An annulus with one boundary component a union of a singular segment and one boundary-orbifold. (2-*pronged crown* and $A(2,2;)$.) It has two corner-reflectors of order 2 if the boundary components are silvered.

(A2) An annulus with one boundary component of the underlying space in a singular locus with one corner-reflector of order n, $n \geq 2$. (The other boundary component is a closed geodesic which is the boundary of the orbifold.) (We call it a *one-pronged crown* and denote it $A(n;)$.)

(A3) A disk with one singular segment and one boundary 1-orbifold and a cone-point of order greater than or equal to three ($D^2(2, 2; n)$).

(A4) A disk with one corner-reflector of order m and one cone-point of order n so that $1/2m + 1/n < 1/2$ (with no boundary orbifold). ($n \geq 3$ necessarily.) ($D^2(m; n)$.)

(D1) A disk with three edges and three boundary 1-orbifolds. No two boundary 1-orbifolds are adjacent. (We call it a *hexagon* or $D^2(2, 2, 2, 2, 2, 2;)$.)

(D2) A disk with three edges and two boundary 1-orbifolds on the boundary of the underlying space. Two boundary 1-orbifolds are not adjacent, and two edges meet in a corner-reflector of order n, and the remaining one a segment. (We called it a *pentagon* and denote it by $D^2(2, 2, 2, 2, n;)$.)

(D3) A disk with two corner-reflectors of order p, q, one of which is greater than or equal to 3, and one boundary 1-orbifold. The singular locus of the disk is a union of three edges and two corner-reflectors. (We call it a *quadrilateral* or $D^2(2, 2, p, q;)$.)

(D4) A disk with three corner-reflectors of order p, q, r where $1/p+1/q+1/r < 1$ and three edges (with no boundary orbifold). (We call it a *triangle* or $D^2(p, q, r;)$.)

The geometric decomposition into elementary orbifolds

- Let Σ be a compact hyperbolic orbifold with $\chi(\Sigma) < 0$ and geodesic boundary.
- Then there exists a mutually disjoint collection of simple closed geodesics and mirror- or cone- or mixed-type geodesic 1-orbifolds so that Σ decomposes along their union to a union of elementary 2-orbifolds or such elementary 2-orbifolds with some boundary 1-orbifolds silvered additionally.

6.5 The Teichmüller spaces for 2-orbifolds

Thurston's theorem

- Let Σ be a compact 2-orbifold with empty boundary and negative Euler characteristic diffeomorphic to an elementary 2-orbifold.
- Then the deformation space $\mathcal{T}(\Sigma)$ of hyperbolic $\mathbb{RP}^2$-structures on Σ is homeomorphic to a cell of dimension $-3\chi(X_\Sigma) + 2k + l + 2n$ where X_Σ is the underlying space and k is the number of cone-points, l is the number of corner-reflectors, and n is the number of boundary full 1-orbifolds.

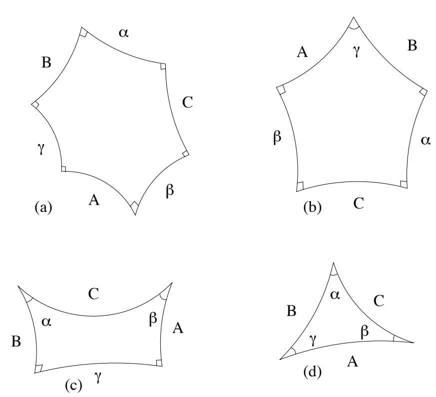
Strategy of proof

- Proposition A: for each elementary 2-orbifold S, $\mathcal{T}(S)$ is homeomorphic to $\mathcal{T}(\partial S)$, where $\mathcal{T}(\partial S)$ is the product of $\mathbb{R}^+$ for each component of ∂S corresponding to the hyperbolic-metric lengths of components of ∂S.
- Then for hyperbolic structures, to obtain a bigger orbifold, we need to use the above result about the Teichmüller spaces under geometric decompositions.

The generalized hyperbolic triangle theorem

- A generalized triangle in the hyperbolic plane is one of following:
  (a) A hexagon: a disk bounded by six geodesic sides meeting in right angles labeled $A, \beta, C, \alpha, B, \gamma$.
  (b) A pentagon: a disk bounded by five geodesic sides labeled A, B, α, C, β where A and B meet in an angle γ, and the rest of the angles are right angles.
  (c) A quadrilateral: a disk bounded by four geodesic sides labeled A, C, B, γ where A and C meet in an angle β, C and B meet in an angle α and the two remaining angles are right angles.
  (d) A triangle: a disk bounded by three geodesic sides labeled A, B, C where A and B meet in an angle γ and B and C meet in an angle α and C and A meet in angle β.

The generalized hyperbolic triangles

- 

The trigonometry

- For generalized triangles in the hyperbolic plane,

$$\begin{aligned}
(\mathrm{a}) \quad \cosh C &= \frac{\cosh\alpha\cosh\beta+\cosh\gamma}{\sinh\alpha\sinh\beta} \\
(\mathrm{b}) \quad \cosh C &= \frac{\cosh\alpha\cosh\beta+\cos\gamma}{\sinh\alpha\sinh\beta} \\
(\mathrm{c}) \quad \sinh A &= \frac{\cosh\gamma\cos\beta+\cos\alpha}{\sinh\beta\sin\gamma} \\
(\mathrm{d}) \quad \cosh C &= \frac{\cos\alpha\cos\beta+\cos\gamma}{\sin\alpha\sin\beta}
\end{aligned} \tag{3}$$

- In (a), (α, β, γ) can be any positive numbers.
- In (b), (α, β) can be any positive numbers and γ in $(0, \pi)$
- In (c), (α, β) can be any positive real numbers in $(0, \pi)$ satisfying $\alpha + \beta < \pi$, and γ any real number.
- In (d), (α, β, γ) can be any real numbers in $(0, \pi)$ satisfying $\alpha + \beta + \gamma < \pi$.

The proof of Proposition A.

- The following lemma implies Proposition A for elementary 2-orbifolds of type (D1), (D2), (D3), and (D4).
- Silvered edges labeled by the capital letters A, B, C. Assign to each vertex an angle of the form π/n (where ($n > 1$ is an integer), for which it is a corner-reflector of that angle. Each edge labeled by Greek letters α, β, γ is a boundary full 1-orbifold.
- Then in cases (a), (b), (c), (d) $\mathcal{F} : \mathcal{T}(P) \to \mathcal{T}(\partial P)$ for each of the above orbifolds P is a homeomorphism; that is, $\mathcal{T}(P)$ is homeomorphic to a cell of dimension 3, 2, 1, or 0 respectively.
- Let S be an elementary 2-orbifold of type (A1), (A2), (A3), or (A4).
- Then $\mathcal{F} : \mathcal{T}(S) \to \mathcal{T}(\partial S)$ is a homeomorphism. Thus, $\mathcal{T}(S)$ is a cell of dimension $2, 1, 1$, or 0 when S is of type (A1), (A2), (A3) or (A4) respectively. In case (A4), $\mathcal{T}(S)$ is a single point.
- For elementary orbifolds of type (P1),(P2),(P3), or (P4), we simply notices that they double covers orbifolds of type (D1),(D2),(D3), or (D4) which is realized as isometries where each of the boundary components do the same. In fact, the isometry can be explictly constructed by taking shortest geodesics between boundary components.

The steps to prove Thurston's theorem.

- Let a 2-orbifold Σ, each component of which has negative Euler characteristic, be in a class $\mathcal{P}$ if the following hold:
  (i) The deformation space of hyperbolic $\mathbb{RP}^2$-structures $\mathcal{T}(\Sigma)$ is diffeomorphic to a cell of dimension
  $$-3\chi(X_\Sigma) + 2k + l + 2n$$
  where k is the number of cone-points, l the number of corner-reflectors, n is the number of boundary full 1-orbifolds.
  (ii) There exists a principal fibration
  $$\mathcal{F} : \mathcal{T}(\Sigma) \to \mathcal{T}(\partial\Sigma)$$
  with the action by a cell of dimension $\dim \mathcal{T}(\Sigma) - \dim \mathcal{T}(\partial\Sigma)$.
- Let Σ be a 2-orbifold whose components are orbifolds of negative Euler characteristic, and it splits into an orbifold Σ' in $\mathcal{P}$.
- We suppose that (i) and (ii) hold for Σ', and show that (i) and (ii) hold for Σ. Since Σ eventually decomposes into a union of elementary 2-orbifolds where (i) and (ii) hold, we would have completed the proof.
- The proof follows by going through each of the constructions....

(A)(I)(1) Let the 2-orbifold Σ'' be obtained from pasting along two closed curves b, b' in a 2-orbifold Σ'. The map resulting from splitting
$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \Delta \subset \mathcal{T}(\Sigma')$$
is a principal $\mathbb{R}$-fibration, where Δ is the subset of $\mathcal{C}(\Sigma')$ where b and b' have equal invariants.

(A)(I)(2) Let Σ'' be obtained from Σ' by cross-capping. The resulting map
$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$
is a diffeomorphism.

(A)(II)(1) Let Σ'' be obtained from Σ' by silvering. The clarifying map
$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$
is a diffeomorphism.

(A)(II)(2) Let Σ'' be obtained from Σ' by folding a boundary closed curve l'. The unfolding map
$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$
is a principal $\mathbb{R}$-fibration.

(B)(I) Let Σ'' be obtained by pasting along two full 1-orbifolds b and b' in Σ'. The splitting map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \Delta \subset \mathcal{T}(\Sigma')$$

is a diffeomorphism where Δ is a subset of $\mathcal{T}(\Sigma')$ where the invariants of b and b' are equal.

(B)(II) Let Σ'' be obtained by silvering or folding a full 1-orbifold. The clarifying or unfolding map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$

is a diffeomorphism.

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- Helpful preliminary knowledge:
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