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THE CONVEX AND CONCAVE DECOMPOSITION OF MANIFOLDS WITH REAL PROJECTIVE STRUCTURES

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Abstract. — We try to understand the geometric properties of n-manifolds ($n \geq 2$) with geometric structures modeled on $(\mathbf{R}P^n, \mathrm{PGL}(n+1, \mathbf{R}))$, i.e., n-manifolds with projectively flat torsion free affine connections. We define the notion of i-convexity of such manifolds due to Carrière for integers i, $1 \leq i \leq n-1$, which are generalization of convexity. Given a real projective n-manifold M, we show that the failure of an $(n-1)$-convexity of M implies an existence of a certain geometric object, n-crescent, in the completion $\check{M}$ of the universal cover $\tilde{M}$ of M. We show that this further implies the existence of a particular type of affine submanifold in M and give a natural decomposition of M into simpler real projective manifolds, some of which are $(n-1)$-convex and others are affine, more specifically concave affine. We feel that it is useful to have such decomposition particularly in dimension three. Our result will later aid us to study the geometric and topological properties of radiant affine 3-manifolds leading to their classification. We get a consequence for affine Lie groups.

Résumé **(La Décomposition conves et concaves de variétés avec les structures projective réelle)**. — Nous essayons de comprendre les propriétés géométriques de n-variétés ($n \geq 2$) avec les structures géomeétriques modelées sur $(\mathbf{R}P^n, \mathrm{PGL}(n+1, \mathbf{R}))$, i.e., les varétés de dimension n avec les connexions affines dont la courbure et la torsion sont nulles. Nous définissons la notion de i-convexité de telles variétés due à Carrière pour les entiers i, $1 \leq i \leq n-1$, qui sont la généralisation de convexité. Etant donné une n-variétés projective réelle M, nous essayons de montrer le fait que léchec d'une $(n-1)$-convexité de M implique une existence d'un certain objet géométrique, n-croissant, dans le complètement $\check{M}$ du revêtement universel $\tilde{M}$ de M. Nous prouvons aussi que cela implique encore l'existence d'un type particulier de subvarété dans M et donne une décomposition naturaelle de M en variétés projectives réelles plus simples, quelques-unes d'entre elles sont $(n-1)$-convexes et les autres sont affines, plus précisément affines concaves. Nous jugeons quíl est utile d'avoir une telle décomposition, en particulier à trois dimensions. Notre résultat aidersa plus tard à étudier les propriétés géometriques et topologiques de 3-variétés affines radiales qui mène à leur classification. Nous obtenons une conséquence pour les groupes Lie affines.

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PREFACE

The purpose of this monograph is to give a self-contained exposition on recent results in flat real projective structures on manifolds. The main result is that manifolds with such structures have canonical geometric decomposition to manifolds with more structures, i.e., ones with better convexity properties and ones which have affine structures of special types.

We hope that this book exposes some of the newly found materials in real projective structures so that more people might become interested in this topic. For that purpose, we include many details missing from previous papers and try to show that the techniques of this paper are at an elementary level requiring only some visualization in spherical and real projective geometry.

Presently, the global study of real projective structures on manifolds is a field which needs to mature with various relevant tools to be discovered. The fact that such geometric structures do not have metrics and such manifolds are often incomplete creates much confusion. Also, as these structures are often assembled in extremely complicated manner as can be seen from their complicated global charts, or developing maps, the manifolds with such structures cannot be seen as having covers in subsets of model geometric spaces. This means that the arguments must be somewhat delicate.

Let us state some reasons why we are interested in real projective structures: Firstly, our decomposition will be helpful for the study of 3-manifolds with flat real projective structures. Already, the theory helps us in classification of radiant affine 3-manifolds (see [**14**]).

All eight of homogeneous 3-dimensional Riemannian geometries can be seen as manifestations of projective geometries, as observed by Thurston. More precisely, Euclidean, spherical, and hyperbolic geometries have projective models. The same can be said of **Sol**-, **Nil**-, and $\widetilde{\mathrm{SL}}(2,\mathbf{R})$-geometries. $H^2 \times \mathbf{S}^1$- and $\mathbf{S}^2 \times \mathbf{S}^1$-geometries are modeled on $\mathbf{R}P^2 \times \mathbf{R}P^1$. Hence, every 3-manifold with homogeneous Riemannian structure has a natural real projective structure or a product real projective structure. One could conjecture that many 3-manifolds admit real projective structures although we do not even have a clue how to go about studying such a question.

Classical affine and projective geometries have plethora of beautiful results giving much insight into Euclidean, spherical, and hyperbolic geometries. We expect that such classical theorems will have important roles to play in the global study of projective structures on manifolds although in the present paper only very small portion of classical geometry is ever used.

As we collect more results on various geometric structures on manifolds, we may gain more perspectives on topology of manifolds which are not available from studying relatively better understood Riemannian homogenous geometric structures. By examining more flexible geometric structures such as foliation, symplectic, contact, conformal, affine, or real projective structures, we may gain more informations about the nature of geometric structures and manifolds in general. (We note here that the comparative study of the geometric structures still have not been delved into much.)

The author thanks Bill Thurston who initiated me into this subject which has much beauty, Bill Goldman who had pioneered some early successful results in this field, Yves Carrière who posed many interesting questions with respect to affine structures, and Hyuk Kim for many sharp observations which helped me to think more clearly. The author benefited greatly from conversations with Thierry Barbot, who suggested the words "convex and concave decomposition", Yves Benoist, Richard Bishop, Craig Hodgson, Michael Kapovich, Steven Kerckhoff, Sadayshi Kojima, François Labourie, Kyung Bai Lee, John Millson, and Frank Raymond. The author thanks the Global Analysis Research Center for generous support and allowing me to enjoy doing mathematics at my slow and inefficient pace.

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Suhyoung Choi
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PART I

AN INTRODUCTION TO REAL PROJECTIVE STRUCTURES

CHAPTER 1

INTRODUCTION

From Ehresmann's definition of geometric structures on manifolds, a *real projective structure* on a manifold is given by a maximal atlas of charts to $\mathbf{R}P^n$ with transition functions extending to projective transformations. (For convenience, we will assume that the dimension n of manifolds is greater that or equal to 2 throughout this paper unless stated otherwise.) This device lifts the real projective geometry locally and consistently on a manifold. In differential geometry, a real projective structure is defined as a projectively flat torsion-free connection. Another equivalent way to define a real projective structure on a manifold M is to give an immersion $\mathbf{dev} : \tilde{M} \rightarrow \mathbf{R}P^n$, a so-called a developing map, equivariant with respect to a so-called holonomy homomorphism $h : \pi_1(M) \rightarrow \mathrm{PGL}(n+1, \mathbf{R})$ where $\pi_1(M)$ is the group of deck transformations of the universal cover $\tilde{M}$ of M and $\mathrm{PGL}(n+1, \mathbf{R})$ is the group of projective transformations of $\mathbf{R}P^n$. (The pair $(\mathbf{dev}, h)$ is said to be the *development pair*.) Each of these descriptions of a real projective structure gives rise to a description of the other two kinds unique up to some natural equivalences.

The global geometric and topological properties of real projective manifolds are completely unknown, and are thought to be very complicated. The study of real projective structure is a fairly obscure area with only handful of global results, as it is a very young field with many open questions, however seemingly unsolvable by traditional methods. The complication comes from the fact that many compact real projective manifolds are not geodesically complete, and often the holonomy groups are far from being discrete lattices and thought to be far from being small such as solvable. There are some early indication that this field however offers many challenges for applying linear representations of discrete groups (which are not lattices), group cohomology, classical convex and projective geometry, affine and projective differential geometry, real algebraic geometry, and analysis. (Since we cannot hope to mention them here appropriately, we offer as a reference the Proceedings of Geometry and Topology Conference at Seoul National University in 1997 [**18**].) This area is also an area closely related to the study of affine structures, which are more extensively studied with regard to affine Lie groups.

Every Riemannian hyperbolic manifold admits a canonical real projective structure, via the Klein model of hyperbolic geometry with the hyperbolic space embedded as the interior of a standard ball in $\mathbf{R}P^n$ and the isometry group $\mathrm{PSO}(1,n)$ as a subgroup of the group $\mathrm{PGL}(n+1,\mathbf{R})$ of projective automorphisms of $\mathbf{R}P^n$ (see [**22**] and [**10**]).

They belong to the class of particularly understandable real projective manifolds which are convex ones. A typical convex real projective manifold is often a quotient of a convex domain in an affine patch of $\mathbf{R}P^n$, i.e., the complement of a codimension one subspace with the natural affine structure of a complete affine space $\mathbf{R}^n$, by a properly discontinuous and free action of a group of real projective transformations (see Theorem 4.5). It admits a Finsler metric, called Hilbert metric, which has many nice geometric properties of a negatively curved Riemannian manifold though the curvature may not be bounded in the sense of Alexandrov (see [**24**]).

An affine structure on a manifold is given by a maximal atlas of charts to an affine space with transition functions affine transformations. Affine manifolds naturally admit a canonical real projective structure since an affine space is canonically identified with the complement of codimension one subspace in the real projective space $\mathbf{R}P^n$ and affine automorphisms are projective. In particular, Euclidean manifolds are projective. Of course there are many affine manifolds which do not come from Euclidean manifolds and most affine structures on manifolds are expected to be not convex (see the classification of affine structures on tori by Nagano-Yagi [**27**]).

Not all real projective manifolds are convex (see [**31**] and [**22**]). However, in dimension two, we showed that closed real projective manifolds are built from convex surfaces. That is, a compact real projective surface of negative Euler characteristic with geodesic boundary or empty boundary decomposes along simple closed geodesics into convex surfaces (see [**10**], [**11**], and [**13**]). With Goldman's classification of convex real projective structures on surfaces [**22**], we obtain a classification of all real projective structures on surfaces [**17**].

Also, recently, Benoist [**5**] classified all real projective structures, left-invariant or not, on nilmanifolds some of which are not convex. Again, the decomposition into parts admitting homogeneous structures was the central results. His student Dupont [**19**] classifies real projective structures on 3-manifolds equal to $\mathbf{Sol}/\Gamma$ for where $\mathbf{Sol}$ is a 3-dimensional solvable Lie group and Γ is a cocompact discrete subgroup.

The real projective structures on 3-manifolds are unexplored area, which may give us some insights into the topology of 3-manifolds along with hyperbolic and contact structures on 3-manifolds.

Three-dimensional manifolds with one of eight Riemannian homogeneous geometric structures admit canonical real projective structures or product real projective structures, as observed by Thurston: Manifolds with hyperbolic, spherical, or Euclidean structures admit canonical real projective structures since hyperbolic, spherical, and Euclidean geometries can be realized as pairs of open subspaces of the real projective space and subgroups of projective automorphisms of the respective subspaces. Similarly, manifolds with **Sol**-, **Nil**-, and $\widetilde{\mathrm{SL}}(2,\mathbf{R})$-structures admit real projective structures. manifolds with $H^2 \times \mathbf{R}$- and $\mathbf{S}^2 \times \mathbf{R}$-structures have product real projective structures modeled on $\mathbf{R}P^2 \times \mathbf{R}P^1$. (See Theorem A.1 using results of Molnár [**26**].)

Also, given a hyperbolic Dehn surgery space of a hyperbolic knot complement in the 3-sphere, the boundary point is often realized by manifolds with degenerate geometric structures. An interesting question by Hodgson is how to understand the degeneration process by real projective structures perhaps by renormalizing the degeneration by projective maps (see the thesis by Suarez [**30**]).

We might ask whether (i) real projective 3-manifolds decompose into pieces which admit one of the above geometries or (ii) conversely pieces with such geometric structures can be glued into real projective 3-manifolds by perturbations. (These are questions by Thurston raised around 1982.)

Goldman (see [**1**, p. 336]) asked which irreducible (Haken) 3-manifolds admit real projective structure? A very exciting development will come from discovering ways to put real projective structures on 3-manifolds other than from homogeneous Riemannian structures perhaps starting from triangulations of 3-manifolds.

A related question asked by John Nash after his showing that all smooth manifolds admit real algebraic structure is when does a manifold admit a rational structure, i.e., an atlas of charts with transition functions which are real rational functions. Real projective manifolds are rational manifolds with more conditions on the transition functions.

These questions are at the moment very mysterious and there is no evidence that they can be answered at all. This paper initiates some methods to study the question (i). We will decompose real projective n-manifolds into concave affine real projective n-manifolds and $(n-1)$-convex real projective n-manifolds.

In three-dimensional case, our resulting decomposition into 2-convex 3-manifolds and concave affine 3-manifolds often seem to be along totally geodesic surfaces, which hopefully will be essential in 3-manifold topology terminology. Thus, our remaining task is to see if 2-convex real projective 3-manifolds admit nice decomposition or at least nice descriptions.

Our result will be used in the decomposition of radiant affine 3-manifolds, which are 3-manifolds with flat affine structure whose affine holonomy groups fix common points of the affine space (see [**14**]). This is the decomposition in Thurston's sense as these manifolds are shown to be Seifert spaces with Euler number zero. In particular, we will be proving there the Carrière conjecture (see [**9**]) that every radiant affine 3-manifold admits a total section to its radial flow, with the help from Barbot's work [**3**], [**4**] (also see his survey article [**2**]). This will result in the classification of radiant affine 3-manifolds.

Let us state our theorems more precisely. Let T be an $(i+1)$-simplex in an affine space $\mathbf{R}^n$, $i+1<n$, with sides $F_1, F_2, \dots, F_{i+2}$. A real projective manifold is said to be *i-convex* if every real projective immersion

$$T^o \cup F_2 \cup \cdots \cup F_{i+2} \to M$$

extends to one from T itself.

***Theorem 1.1* (Main).** — *Suppose that M is a compact real projective n-manifold with empty or totally geodesic boundary. If M is not $(n-1)$-convex, then M includes a compact concave affine n-submanifold N of type I or II or M^o includes the canonical two-faced $(n-1)$-submanifold of type I or II.*

We will define the term "two-faced $(n-1)$-submanifolds of type I and II" in Definitions 6.5 and 7.7 which arise in separate constructions. But they are totally geodesic and are quotients of open domains in the affine space by groups of projective transformations and are canonically defined. A two-dimensional example with a nontrivial splitting is given in Example 7.9. We define the term concave affine n-submanifold in Definition 9.1: A concave affine n-manifold M is a real projective manifold with concave boundary such that its cover is a union of overlapping n-crescents. The manifold-interior M^o of a concave affine manifold admits a projectively equivalent affine structure of very special nature. We expect them to be very limited. A so-called n-crescent is a convex n-ball whose bounding sides except one is in the "infinity" of the completion of the universal or holonomy cover (see Definition 3.6). Their interiors are projectively diffeomorphic to either a half-space or an open hemisphere. They are really generalization of affine half-spaces as one of the side is at "infinity" or "missing".

Let A be a properly imbedded $(n-1)$-manifold in M^o, which may or may not be two-sided and not necessarily connected or totally geodesic. The so-called splitting S of M along A is obtained by completing $M - N$ by adding boundary which consists of either the union of two disjoint copies of components of A or double covers of components of A (see the beginning of Chapter 10).

A manifold N *decomposes into* manifolds $N_1, N_2, \dots$ if there exists a properly imbedded $(n-1)$-submanifold Σ so that N_i are components of the manifold obtained from splitting M along Σ; $N_1, N_2, \dots$ are said to be the *resulting* manifolds of the decomposition.

Corollary 1.2. — *Let M be a compact real projective n-manifold with empty or totally geodesic boundary. Suppose that M is not $(n-1)$-convex. Then*

1. *after splitting M along the two-faced $(n-1)$-manifold A_1 arising from hemispheric n-crescents, the resulting manifold M^{s} decomposes into compact concave affine manifolds of type I and real projective n-manifolds with totally geodesic boundary which does not include any compact concave affine manifolds of type I.*
2. *We let N be the disjoint union of the resulting manifolds of the above decomposition other than concave affine ones. After cutting N along the two-faced $(n-1)$-manifold A_2 arising from bihedral n-crescents, the resulting manifold N^{s} decomposes into maximal compact concave affine manifolds of type II and real projective n-manifolds with convex boundary which is $(n-1)$-convex and includes no compact concave affine manifold of type II.*

Furthermore, A_1 and A_2 are canonically defined and the decomposition is also canonical in the following sense: *If M^{s} equals $N \cup K$ for K the union of compact concave affine manifolds of type I in M^{s} and N the closure of the complement of K includes* *no compact concave affine manifolds of type I, then the above decomposition agree with the decomposition into components of submanifolds in* (1). *If* N^{s} *equals* $S \cup T$ *for* T *the finite disjoint union of maximal compact concave affine manifolds of type II in* N^{s} *and* S *the closure of the complement of* T *that is* $(n-1)$*-convex and includes no compact concave affine manifold of type II, then the decomposition agrees with the decomposition into components of submanifolds in* (2).

By a *maximal* compact concave affine manifold of type II, we mean one which is not a proper subset of another compact concave affine manifold of type II. If $A_1 = \emptyset$, then we define $M^{\mathrm{s}} = M$ and if $A_2 = \emptyset$, then define $N^{\mathrm{s}} = N$.

We note that M, M^{s}, N, and N^{s} have totally geodesic or empty boundary, as we will see in the proof. The final decomposed pieces of N^{s} are not so. Concave affine manifolds of type II have in general boundary concave seen from their inside and the $(n-1)$-convex real projective manifolds have convex boundary seen from inside (see Chapter 3).

Compare this corollary with what we have proved in [**10**] and [**11**] in the language of this paper, as the term "decomposition" is used somewhat differently there by not allowing one-sided closed geodesics to be used for decomposition.

Theorem 1.3. — *Let* Σ *be a compact real projective surface with totally geodesic or empty boundary. Suppose* $\chi(\Sigma) < 0$. *Then* Σ *decomposes along the union of disjoint simple closed curves into convex real projective surfaces.*

Our Corollary 1.2 is strong enough to imply Theorem 1.3, but we need to work out the classification of concave affine 2-manifolds to do so.

This monograph will be written as self-contained as possible on projective geometry and will use no highly developed machinery but will use perhaps many aspects of discrete group actions and geometric convergence in the Hausdorff sense joined in a rather complicated manner. Objects in this papers are all very concrete ones. To grasp these ideas, one only needs to have some graduate student in geometry understanding and visualization of higher-dimensional projective and spherical geometry. The main methods are extended from those already used in dimension two.

We work on $n \geq 2$ case although $n = 2$ case was more completely answered in the earlier papers [**10**] and [**11**] (see Theorem 1.3). The point where this monograph improves the papers [**10**] and [**11**] even in $n = 2$ case is that we will be introducing the notion of two-faced submanifolds which makes decomposition easier to understand.

A *holonomy cover* of M is given as the cover of M corresponding to the kernel of the holonomy homomorphism. We often need not look at the universal cover but the holonomy cover as it carries all information and we can define the developing map and holonomy homomorphism from it. The so-called Kuiper completion or projective completion of the universal or holonomy cover is the completion with respect to a metric pulled from $\mathbf{S}^n$ by a developing map, as was introduced by Kuiper for conformally flat manifolds (see Kuiper [**25**]).

In Part I, we give an introduction to projective geometry on spheres and the Kuiper completions of real projective manifolds. In Chapter 2, we will give preliminary definitions and define and classify convex sets in $\mathbf{S}^n$. We also discuss the geometric limit of a sequence of convex balls. In Chapter 3, we discuss the Kuiper or projective completions $\check{M}$ or $\check{M}_h$ of the universal cover $\tilde{M}$ or the holonomy cover M_h respectively and convex subsets of them, and discuss how two convex subsets may intersect, showing that in the generic case they can be read from their images in $\mathbf{S}^n$. We also introduce "dipping intersection". This is when we can realize the intersection of two convex balls as the closure of a component of a ball removed with a side of the other ball. We finally discuss the convergence of sequences of convex balls in the Kuiper completions.

In Part II, we will prove main results of this paper. The main focus in this paper is to get good geometric objects in the universal cover of M. Loosely speaking, we illustrate our plan as follows:

(i) For a compact manifold M which is not $(n-1)$-convex, obtain an n-crescent in $\check{M}_h$.
(ii) Divide into two cases where $\check{M}_h$ includes hemispheric n-crescents and where n-crescents are always bihedral.
(iii) We derive a certain equivariance properties of hemispheric n-crescents or the unions of a collection of bihedral n-crescents equivalent to each other under the equivalence relation generated by the overlapping relation. That is, we show that any two of such sets either agree, are disjoint, or meet only in the boundary.
(iv) We show that the boundary where the two collections meet covers a closed codimension-one submanifold called the two-faced submanifolds. If we split M along these, then the collection is now truly equivariant. From the equivariance, we obtain submanifolds covered by them called the concave affine manifolds. This completes the proof of the Main Theorem.
(v) Apply the Main Theorem in sequence to prove Corollary 1.2; that is, we split along the two-faced manifolds and obtain concave affine manifolds for hemispheric n-crescent case and then bihedral n-crescent case.

In Chapter 4, we prove a central theorem that given a real projective manifold which is not $(n-1)$-convex, we can find an n-crescent in the projective completion. The argument is the blowing up or pulling back argument as we saw in [**10**].

In Chapter 5, we generalize the transversal intersection of crescents to that of n-crescents (see [**10**]). This shows that they intersect in a manageable manner so that their sides in the ideal set extend each other and the remaining sides intersecting transversally.

In Chapter 6, when $\check{M}_h$ includes a hemispheric n-crescent, we show how to obtain a two-faced $(n-1)$-submanifold. This is accomplished by the fact that two hemispheric crescents are either disjoint, equal, or meet only at the boundary, i.e., at a totally geodesic $(n-1)$-manifold which covers a closed totally geodesic $(n-1)$-submanifold in M, a so-called two-faced submanifold. In Chapter 7, we assume that $\check{M}_h$ includes no hemispheric n-crescent but includes bihedral n-crescents. We define equivalence classes of bihedral n-crescents. Two bihedral n-crescents are equivalent if there exists a chain of bihedral n-crescents overlapping with the next ones in the chain. This enables us to define $\Lambda(R)$ the union of n-crescents equivalent to a given n-crescent

R. Given $\Lambda(R)$ and $\Lambda(S)$ for two n-crescents R and S, they are either disjoint, equal, or meet at a totally geodesic $(n-1)$-submanifold. We obtain a two-faced $(n-1)$-submanifold from the totally geodesic $(n-1)$-submanifolds.

In Chapter 8, we show what happens to n-crescents if we take submanifolds or splits manifolds in the corresponding completions of the holonomy cover. They are all preserved.

In Chapter 9, we prove the Main Theorem: If there is no two-faced submanifold of type I, then two hemispheric n-crescents are either disjoint or equal. The union of all hemispheric n-crescents left-invariant by deck transformations and hence covers a submanifold in M, a finite disjoint union of compact concave affine manifolds of type I. If there is no two-faced submanifold of type II, then $\Lambda(R)$ and $\Lambda(S)$ for two n-crescents R and S are either disjoint or equal. Again since the deck transformation group acts on the union of $\Lambda(R)$ for all n-crescents R, the union covers a manifold in M, a finite disjoint union of compact concave affine manifolds of type II.

In Chapter 10, we prove Corollary 1.2; we decompose real projective manifolds. We show that when we have a two-faced submanifold, we can cut M along these. The result does not have a two-faced submanifold and hence can be decomposed into $(n-1)$-convex ones and properly concave affine manifolds as in Chapter 9.

In Chapter 11, we will show some consequence or modification of our result for Lie groups with left-invariant real projective or affine structures.

A real projective structure on a Lie group is *left-invariant* if left-multiplications preserve the real projective structure. The methods of the following theorem is also applicable to real projective structures on homogeneous manifolds invariant with respect to proper group actions (see Theorem 11.3).

Theorem 1.4. — *Let G be a Lie group with left-invariant real projective structure. Then either G is $(n-1)$-convex or its universal cover $\tilde G$ is projectively diffeomorphic to the universal cover of the complement of a closed convex set in $\mathbf{R}^n$ with induced real projective structure.*

The $(n-1)$-convexity of affine structures are defined similarly, and this theorem easily translates to one on affine Lie groups:

Corollary 1.5. — *Suppose that G has a left-invariant affine structure. Then either G is $(n-1)$-convex or $\tilde G$ is affinely diffeomorphic to the universal cover of the complement of a closed convex set in $\mathbf{R}^n$ with induced affine structure.*

Part III consists of two appendices: In Appendix A, we show that 3-manifolds with homogeneous Riemannian geometric structures admit canonical real projective structures or product real projective structures using results of Molnár [**26**]. We show that a real projective manifold is convex if and only if it is a quotient of a convex domain in $\mathbf{S}^n$. In Appendix B, we study some questions on shrinking sequences of convex balls in $\mathbf{S}^n$ that are needed in Chapter 4.

CHAPTER 2

CONVEX SUBSETS OF THE REAL PROJECTIVE SPHERE

In this chapter, we will discuss somewhat slowly the real projective geometry of $\mathbf{R}P^n$ and the sphere $\mathbf{S}^n$, and discuss convex subsets of $\mathbf{S}^n$. We will give classification of convex subsets and give topological properties of them. We end with the geometric convergence of convex subsets. (We assume that the reader is familiar with convex sets in affine spaces, which are explained in Berger [**7**] and Eggleston [**20**] in detailed and complete manner.)

The real projective space $\mathbf{R}P^n$ is the quotient space of $\mathbf{R}^{n+1} - \{O\}$ by the equivalence relation $\sim$ given by $x \sim y$ iff $x = sy$ for two nonzero vectors x and y and a nonzero real number s. The group $\mathrm{GL}(n+1, \mathbf{R})$ acts on $\mathbf{R}^{n+1} - \{O\}$ linearly and hence on $\mathbf{R}P^n$, but not effectively. However, the group $\mathrm{PGL}(n+1, \mathbf{R})$ acts on $\mathbf{R}P^n$ effectively. The action is analytic, and hence any element acting trivially in an open set has to be the identity transformation. (We will assume that $n \geq 2$ for convenience.)

Real projective geometry is a study of the invariant properties of the real projective space $\mathbf{R}P^n$ under the action of $\mathrm{PGL}(n+1, \mathbf{R})$. Given an element of $\mathrm{PGL}(n+1, \mathbf{R})$ we identify it with the corresponding projective automorphism of $\mathbf{R}P^n$.

By a real projective manifold, we mean an n-manifold with a maximal atlas of charts to $\mathbf{R}P^n$ where the transition functions are projective. This lifts all local properties of real projective geometry to the manifold. A *real projective map* is an immersion from a real projective n-manifold to another one which is projective under local charts. More precisely, a function $f : M \to N$ for two real projective n-manifolds M and N is *real projective* if it is continuous and for each pair of charts $\phi : U \to \mathbf{R}P^n$ for M and $\psi : V \to \mathbf{R}P^n$ for N such that U and $f^{-1}(V)$ overlap, the function

$$\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(f(U) \cap V)$$

is a restriction of an element of $\mathrm{PGL}(n+1, \mathbf{R})$ (see Ratcliff [**28**]).

It will be very convenient to work on the simply connected sphere $\mathbf{S}^n$ the double cover of $\mathbf{R}P^n$ as $\mathbf{S}^n$ is orientable and it is easier to study convex sets. We may identify the standard unit sphere $\mathbf{S}^n$ in $\mathbf{R}^{n+1}$ with the quotient space of $\mathbf{R}^{n+1} - \{O\}$ by the equivalence relation $\sim$ given by $x \sim y$ if $x = sy$ for nonzero vectors x and y and $s > 0$.

As above $\mathrm{GL}(n+1, \mathbf{R})$ acts on $\mathbf{S}^n$. The subgroup $\mathrm{SL}_{\pm}(n+1, \mathbf{R})$ of linear maps of determinant ± 1 acts on $\mathbf{S}^n$ effectively. We see easily that $\mathrm{SL}_{\pm}(n+1, \mathbf{R})$ is a double cover of $\mathrm{PGL}(n+1, \mathbf{R})$. We denote by $\mathrm{Aut}(\mathbf{S}^n)$ the isomorphic group of automorphisms of $\mathbf{S}^n$ induced by elements of $\mathrm{SL}_{\pm}(n+1, \mathbf{R})$.

Since $\mathbf{R}P^n$ has an obvious chart to itself, namely the identity map, it has a maximal atlas containing this chart. Hence, $\mathbf{R}P^n$ has a real projective structure. Since $\mathbf{S}^n$ is a double cover of $\mathbf{R}P^n$, and the covering map p is a local diffeomorphism, it follows that $\mathbf{S}^n$ has a real projective structure. $\mathbf{S}^n$ with this canonical real projective structure is said to be a ***real projective sphere***. We see easily that each element of $\mathrm{Aut}(\mathbf{S}^n)$ are real projective maps. Conversely, each real projective automorphism of $\mathbf{S}^n$ is an element of $\mathrm{Aut}(\mathbf{S}^n)$ as the actions are locally identical with those of elements of $\mathrm{Aut}(\mathbf{S}^n)$. There is a following convenient commutative diagram:

$$\begin{array}{ccc} \mathbf{S}^n & \xrightarrow{g} & \mathbf{S}^n \\ \downarrow p & & \downarrow p \\ \mathbf{R}P^n & \xrightarrow{g'} & \mathbf{R}P^n \end{array} \tag{2.1}$$

where given a real projective automorphism g, a real projective map g' always exists and given g', we may obtain g unique up to the antipodal map $A_{\mathbf{S}^n}$ which sends x to its antipodal point x^- for each unit vector x in $\mathbf{S}^n$.

The standard sphere has a standard Riemannian metric μ of curvature 1. We denote by $\mathbf{d}$ the path-metric on $\mathbf{S}^n$ induced from μ. The geodesics of this metric are paths on a great circles parameterized by $\mathbf{d}$-length. This metric is projectively flat, and hence geodesics of the metric agree with projective geodesics up to choices of parameterization.

A *convex line* is an embedded geodesic in $\mathbf{S}^n$ of $\mathbf{d}$-length less than or equal to π. A *convex subset* of $\mathbf{S}^n$ is a subset such that any two points of A are connected by a convex segment in A. A *simply convex* subset of $\mathbf{S}^n$ is a convex subset such that every pair of points are connected by a convex segment of $\mathbf{d}$-length $< \pi - \epsilon$ for a positive number ϵ. (Note that all these are projectively invariant properties.) A singleton, i.e., the set consisting of a point, is convex and simply convex.

A *great* 0*-dimensional sphere* is the set of points antipodal to each other. This is not convex. A *great i-dimensional sphere* in $\mathbf{S}^n$ for $i \geq 1$ is convex but not simply convex. An *i-dimensional hemisphere*, $i \geq 1$, is the closure of a component of a great i-sphere $\mathbf{S}^i$ removed with a great $(i-1)$-sphere $\mathbf{S}^{i-1}$ in $\mathbf{S}^i$. It is a convex but not simply convex. A 0*-dimensional hemisphere* is simply a singleton.

Given a codimension one subspace $\mathbf{R}P^{n-1}$ of $\mathbf{R}P^n$, the complement of $\mathbf{R}P^n$ can be identified with an affine space $\mathbf{R}^n$ so that geodesic structures agree, i.e., the projective geodesics are affine ones and vice versa up to parameterization. Given an affine space $\mathbf{R}^n$, we can compactify it to a real projective space $\mathbf{R}P^n$ by adding points (see Berger [**7**]). Hence the complement $\mathbf{R}P^n - \mathbf{R}P^{n-1}$ is called an *affine patch*. An open n-hemisphere in $\mathbf{S}^n$ maps homeomorphic onto $\mathbf{R}P^n - \mathbf{R}P^{n-1}$ for a subspace $\mathbf{R}P^{n-1}$. Hence, the open n-hemisphere has a natural affine structure of $\mathbf{R}^n$ whose geodesic structure is same as that of the projective structure. An open n-hemisphere is sometimes called an *affine patch*.

A subset of $\mathbf{R}^n$ convex in the affine sense is convex in $\mathbf{S}^n$ by our definition when $\mathbf{R}^n$ is identified with the open n-hemisphere in this manner.

We give a definition given in [**28**]: A pair of points x and y is *proper* if they are not antipodal. A minor geodesic connecting a proper pair x and y is the shorter path in the great circle passing through x and y with boundary x and y.

The following proposition shows the equivalence of our definition to one given in [**28**] except for pairs of antipodal points.

Proposition 2.1. — *A set A is a convex set or a pair of antipodal points if and only if for each proper pair of points x, y in A, A includes a minor geodesic $\overline{xy}$ in A connecting x and y.*

Proof. — If A is convex, then given two proper pair of points the convex segment in A connecting them is clearly a minor geodesic. A pair of antipodal points has no proper pair.

Conversely, let x and y be two points of A. If x and y are proper then since a minor geodesic is convex, we are done. If x and y are antipodal, and A equals $\{x, y\}$, then we are done. If x and y are antipodal, and there exists a point z in A distinct from x and y, then A includes the minor segment $\overline{xz}$ and $\overline{yz}$ and hence $\overline{xz} \cup \overline{yz}$ is a convex segment connecting x and y; A is convex. $\square$

By the above proposition, we see that our convex sets satisfy the properties in Section 6.2 of [**28**]. Let A be a nonempty convex subset of $\mathbf{S}^n$. The *dimension* of A is defined to be the least integer m such that A is included in a great m-sphere in $\mathbf{S}^n$. If $\dim(A) = m$, then A is included in a unique great m-sphere which we denote by $\langle A \rangle$. The interior of A, denoted by A^o, is the topological interior of A in $\langle A \rangle$, and the boundary of A, denoted by ∂A, is the topological boundary of A in $\langle A \rangle$. The closure of A is denoted by $\mathrm{Cl}(A)$ and is a subset of $\langle A \rangle$. $\mathrm{Cl}(A)$ is convex and so is A^o. Moreover, the intersection of two convex sets is either convex or is a pair of antipodal points by the above proposition. Hence, the intersection of two convex sets is convex if it contains at least three points, it contains a pair of nonantipodal points, or one of the sets contains no pair of antipodal points.

A *convex hull* of a set A is the minimal convex set including A. A *side* of a convex set A is a maximal convex subset of ∂A. A *polyhedron* is a convex set with finitely many sides.

Lemma 2.2. — *Let A be a convex set. A^o is not empty unless A is empty.*

Proof. — Let $\langle A \rangle$ have dimension k. Then A has to have at least $k+1$ points $p_1, \dots, p_{k+1}$ in general position as unit vectors in $\mathbf{R}^{n+1}$ since otherwise every $(k+1)$-tuple of vectors are dependent and A is a subset of a great sphere of lower dimension. The convex hull of the points $p_1, \dots, p_{k+1}$ is easily shown to be a spherical simplex with vertices $p_1, \dots, p_{k+1}$. The simplex is obviously a subset of A, and the interior of the simplex is included in A^o. $\square$

We give the classification of convex sets in the following two propositions.

Proposition 2.3. — *Let A be a convex subset of $\mathbf{S}^n$. Then A is one of the following sets*:

1. *a great sphere $\mathbf{S}^m$, $1 \leq m \leq n$,*
2. *an m-dimensional hemisphere H^m, $0 \leq m \leq n$,*
3. *a proper convex subset of an i-hemisphere H^m.*

Proof. — We will prove by induction on dimension m of $\langle A \rangle$. The theorem is obvious for $m = 0, 1$. Suppose that the theorem holds for $m = k-1$, $k \geq 2$. Suppose now that the dimension of A equals m for $m = k$. Let us choose a hypersphere $\mathbf{S}^{m-1}$ in $\langle A \rangle$ intersecting with A^o. Then $A_1 = A \cap \mathbf{S}^{m-1}$ is as one of the above (1), (2), (3). The dimension of A_1 is at least one, i.e., $m - 1 \geq 1$. Suppose $A_1 = \mathbf{S}^{m-1}$. As A^o has two points x, y respectively in components of $\langle A \rangle - \mathbf{S}^{m-1}$, taking the union of segments from x to points of $\mathbf{S}^{m-1}$, and segments from y to points of $\mathbf{S}^{m-1}$, we obtain that $A = \langle A \rangle$.

If A_1 is as in (2) or (3), then choose an $(m-1)$-hemisphere H including A_1 with boundary a great $(m-2)$-sphere ∂H. Consider the collection $\mathcal{P}$ of all $(m-1)$-hemispheres including ∂H. Then $\mathcal{P}$ has a natural real projective structure of a great circle, and let A' be the set of the $(m-1)$-hemispheres in $\mathcal{P}$ whose interior meets A. Then since a convex segment in $\langle A \rangle - \partial H$ projects to a convex segment in the circle $\mathcal{P}$, it follows that A' has the property that any proper pair of points of A' is connected by a minor geodesic, and by Proposition 2.1 A' is either a pair of antipodal points or a convex subset.

Let H^- denote the closure of the complement of $\langle A_1 \rangle - H$. Then the interior of H^- do not meet A as it does not meet A_1. Hence A' is a subset of $\mathcal{P} - \{H^-\}$.

If A' is a pair of antipodal points, then A' must be $\{H, H^-\}$, and this is a contradiction. Since A' is a proper convex subset of $\mathcal{P}$, A' must be a convex subset of a 1-hemisphere I in $\mathcal{P}$. This means that only the interior of $(m-1)$-hemispheres in I meets A, and there exists an m-hemisphere in $\langle A \rangle$ including A. Thus A either equals this m-hemisphere or a proper convex subset of it. □

We say that a subset of a real projective manifold satisfies the *Kobayashi's criterion* if there is no non-constant projective map from the real line $\mathbf{R}$ to it. (A convex open domain in $\mathbf{S}^n$ satisfying Kobayashi's criterion has a complete Hilbert metric by Proposition 3.20 [**24**].)

Given a convex compact subset A of $\mathbf{S}^n$ the following statements are equivalent:

- A satisfies Kobayashi's criterion.
- A does not include a line whose $\mathbf{d}$-length equals π.
- A does not include a pair of points antipodal to each other.

Let A be a convex subset of an i-hemisphere H^i for $i \geq 1$. Assume that A does not satisfy Kobayashi's criterion. Then there are great spheres of dimension ≥ 0 in $\mathrm{Cl}(A)$. Since great spheres in H^i are subsets of ∂H^i, they are included in $\partial A \cap \partial H^i$.

Given two great spheres $\mathbf{S}^j$ and $\mathbf{S}^k$ in A for $0 \leq j, k \leq i-1$, the geometry of H^i and the convexity of A easily imply that there exists a great sphere $\mathbf{S}^l$ with $j, k < l \leq i-1$ including the both spheres. (Note that if $k = i-1$, then A equals H^i.)

Proposition 2.4. — *Let A be a convex subset of an i-dimensional hemisphere H^i for $i \geq 1$. Then exactly one of the following holds:*

- *∂A includes a unique maximal great j-sphere $\mathbf{S}^j$ for some $0 \leq j \leq i-1$, which must be in ∂H^i and the closure of A is the union of $(j+1)$-hemispheres with common boundary $\mathbf{S}^j$, or*
- *A is a simply convex subset of H^i, in which case A can be realized as a bounded convex subset of perhaps another open i-hemisphere K^i identified with an affine space $\mathbf{R}^i$.*

Proof. — We assume without loss of generality that A is closed by taking the closure of A if necessary. The first item is proved in the above paragraph since the second statement of the first item simply follows from convexity of A.

If A includes no pair of antipodal points, then let m be the dimension of $\langle A\rangle$ and we do the induction over m. If $m = 0, 1$, then the second item is obvious. Suppose we have the second item holding for $m = k-1$, where $k \geq 2$. Now let $m = k$, and choose a great sphere $\mathbf{S}^{m-1}$ meeting A^o, and let $A_1 = A \cap \mathbf{S}^{m-1}$. Since A_1 is another simply convex set, A_1 is a bounded convex subset of an open $(m-1)$-hemisphere K identified as an affine space $\mathbf{R}^{m-1}$. Hence A_1 does not meet ∂K. As in the proof of Proposition 2.3, we let $\mathcal{P}$ be the set of all $(m-1)$-hemispheres with boundary in ∂K, which has a natural real projective structure of a great circle. As in the proof, we see that the subset A' of $\mathcal{P}$ consisting of hemispheres whose interior meets A is a convex subset of a 1-hemisphere in $\mathcal{P}$. The boundary of A' consists of two hemispheres H_1 and H_2. Since A' is connected, H_1 and H_2 bound a convex subset L in $\langle A\rangle$, and H_1 and H_2 meet in a μ-angle less than or equal to π.

If the angle between H_1 and H_2 equals π, then $H_1 \cup H_2$ is a great $(m-1)$-sphere, and H_1^o and H_2^o contains two points p, q of A respectively which are not antipodal. Since A is convex, $\overline{pq}$ is a subset of A; since p and q is not antipodal, $\overline{pq}$ meets ∂K by geometry, a contradiction.

Since the angle between H_1 and H_2 is less then π, it is now obvious that there exists an m-hemisphere H including A and meeting L only at ∂K. Hence A is a convex subset of H^o. Since A is compact, A is a bounded convex subset of H^o. $\square$

An *m-bihedron* in $\mathbf{S}^n$ is the closure of a component of a great sphere $\mathbf{S}^m$ removed with two great spheres of dimension $m-1$ in $\mathbf{S}^m$ ($m \geq 1$). A 1-bihedron is a simply convex segment.

Lemma 2.5. — *A compact convex subset K of $\mathbf{S}^n$ including an $(n-1)$-hemisphere is either the sphere $\mathbf{S}^n$, a great $(n-1)$-sphere, an n-hemisphere, an n-bihedron, or the $(n-1)$-hemisphere itself.*

Proof. — Let H be the $(n-1)$-hemisphere in K and s the great circle perpendicular to H at the center of H. Then since K is convex, $s \cap K$ is a convex subset of s or a pair of antipodal point as in the proof of Proposition 2.3. If $s \cap K = s$, then every segment from a point of s to a point of H belongs to K by convexity. Thus, $K = \mathbf{S}^n$. Depending on whether $s \cap K$ is a point, a pair of antipodal points, a 1-hemisphere or a simply convex segment, K is H, a great $(n-1)$-sphere, an n-hemisphere or an n-bihedron. □

Proposition 2.6. — *Let A be a convex m-dimensional subset of $\mathbf{S}^n$ other than a great sphere. Then A^o is homeomorphic to an open m-ball, $\mathrm{Cl}(A)$ the compact m-ball, and ∂A to the sphere of dimension $m-1$.*

Proof. — We can generalize Section 11.3.1 of Berger [**7**] to prove this proposition. □

Let A be an arbitrary subset of $\mathbf{S}^n$ and x a point of the topological boundary $\mathrm{bd}A$ of A. A *supporting* hypersphere L for A is a great $(n-1)$-sphere containing x in A such that the two closed hemispheres determined by L includes A and x respectively. We say that L is the *supporting* hypersphere for A at x.

Proposition 2.7. — *Let A be a convex subset of $\mathbf{S}^n$, other than $\mathbf{S}^n$ itself. Then for each point x of ∂A, there exists a supporting hypersphere for A at x.*

Proof. — If the dimension i of A is 0, this is trivial. Assume $i \geq 1$. If A is a great i-sphere or an i-hemisphere $i \geq 1$, it is obvious. If not, then A is included an i-hemisphere, say H. Then A^o is a convex subset of the affine space H^o. If $x \in H^o$, there exists a supporting hyperplane K for A^o at x by Proposition 11.5.2 of [**7**]. The hyperplane K equals $L \cap H^o$ for a great $(i-1)$-sphere L in $\langle A \rangle$. Thus any great $(n-1)$ sphere P meeting $\langle A \rangle$ at L is the supporting hypersphere for A at x. If $x \in \partial H$, then the conclusion is obvious. □

We define the Hausdorff distance $\mathbf{d}^H$ between all compact subsets of $\mathbf{S}^n$: We say that two compact subsets X, Y have distance $\mathbf{d}^H$ less than ϵ, if X is in an ϵ-$\mathbf{d}$-neighborhood of Y and Y is in one of X. This defines a metric on the space of all compact subsets of $\mathbf{S}^n$.

Suppose that a sequence of compact sets K_i converges to K_∞. Then it is well-known that x belongs to K_∞ if and only if x is a limit of a sequence $\{x_i\}$, $x_i \in K_i$: If $x \in K_\infty$, then by definition for any positive number ϵ, there exists an N so that for $i > N$, K_i contains a point x_i so that $\mathbf{d}(x, x_i) < \epsilon$. Also, given a point x of $\mathbf{S}^n$, if a sequence $x_i \in K_i$ converges to x, then x lies in K_∞. If otherwise, x is at least δ away from K_∞ for $\delta > 0$, and so the $\delta/2$-$\mathbf{d}$-neighborhood of K_∞ is disjoint from an open neighborhood J of x. But since $x_i \in J$ for i sufficiently large, this contradicts $K_i \to K_\infty$.

Proposition 2.8. — *Given a sequence of compact convex subsets K_i of $\mathbf{S}^n$, we can always choose a subsequence converging to a subset K_∞. K_∞ is compact and convex. Also the following hold*:

- *If K_i are great i-spheres, then K_∞ is a great i-sphere.*
- *If K_i are i-hemispheres, that K_∞ is an i-hemisphere.*
- *If K_i are i-bihedrons, then K_∞ is either an i-hemisphere, an i-bihedron, or an $(i-1)$-hemisphere.*
- *If K_i are i-balls, then K_∞ is a convex ball of dimension less than or equal to i.*

Proof. — The first statement follows from the well-known compactness of the spaces of compact subsets of compact metric spaces under Hausdorff metrics.

For each point x of K_∞, there exists a sequence $x_i \in K_i$ converging to x. Choose arbitrary two distinct points x and y of K_∞, and sequences $x_i \in K_i$ and $y_i \in K_i$ converging to x and y respectively. Then there exists a segment $\overline{x_i y_i}$ of $\mathbf{d}$-length $\leq \pi$ in K_i connecting x_i and y_i. Since the sequence of $\overline{x_i y_i}$ is a sequence of compact subsets of $\mathbf{S}^n$, we may assume that a subsequence converges to a compact subset L of $\mathbf{S}^n$. By the above paragraph $L \subset K_\infty$. Since 1-bihedrons and 1-hemispheres are nothing but convex segments, the second and third items imply that L is a convex segment. Thus K_∞ is convex.

(1) A great i-sphere is defined by $n - i$ number of dual $\mathbf{d}$-orthonormal vectors of $\mathbf{R}^n$. Let $\{s_1^i, \dots, s_{n-k}^i\}$ for $i = 1, 2, \dots$ to be the set of dual vectors for a great sphere K_i. Then a point x belongs to K_∞ if and only if it is a limit of a sequence of points $x_i \in K_i$. Hence, x belongs to K_∞ if and only if x is zero under the set of limit dual vectors. Hence, K_∞ is precisely defined by a set of $(n - i)$-equations and is a great i-sphere.

(2) This follows as in (1) using $\mathbf{d}$-orthonormal dual vectors defining an i-hemisphere.

(3) An i-bihedron is defined by $n - i$ $\mathbf{d}$-orthonormal vectors defining the great i-sphere including it and two $\mathbf{d}$-unit vectors which are normal to the $n - i$ vectors but may have an angle with respect to each other.

(4) If K_i are i-balls, then $K_i \subset H_i$ for i-hemispheres H_i. We choose a subsequence i_j of i so that H_{i_j} converges to an i-hemisphere H. It follows that K_∞ is a subset of H by the paragraph above our proposition since K_{i_j} converges to K_∞. Thus, K_∞ is a compact convex subset of H, which shows that K_∞ is a convex ball of dimension $\leq i$ by Proposition 2.6. □

Remark 2.9. — Contrary to above a sequence of simply convex i-balls can converge to an i-hemisphere or nonsimply convex i-balls. Given a sequence of i-balls K_i, if L_i is the sequence of maximal great spheres in K_i of dimension j_i, then the limit K_∞ includes the limits of subsequences of L_i and the maximal great sphere for K_∞ has dimension greater than or equal to the limit supremum of the sequence of the dimensions of L_i.

Proposition 2.10. — *Let K_i be a sequence of convex n-balls. If $\dim K_\infty = n$, then we have $\bigcup_{i=1}^\infty K_i^o \supset K_i^o$. In this case $\partial K_i \to \partial K_i$.*

The proof follows as in Section 2 of Appendix of [**10**]. The dimension does not play a role. We will give a shortened proof here for reader's convenience.

Given a great sphere $\mathbf{S}^{n-1}$ in $\mathbf{S}^n$ and a point x belonging to $\mathbf{S}^n - \mathbf{S}^{n-1}$, if a geodesic from x to $\mathbf{S}^{n-1}$ is perpendicular to $\mathbf{S}^{n-1}$, and its $\mathbf{d}$-length $\leq \pi/2$, then its $\mathbf{d}$-length equals $\mathbf{d}(x, \mathbf{S}^{n-1})$.

Suppose that $x \in B$ for a convex i-ball B. Then we have $\mathbf{d}(x, \partial B) \leq \pi/2$. We have $\mathbf{d}(x, \partial B) = \pi/2$ if and only if B is an i-hemisphere of which x is the center.

Lemma 2.11. — *Let A and B be two convex n-balls in $\mathbf{S}^n$. Suppose that $A^o - B^o$ contains a point x such that $\mathbf{d}(x, \partial A) > 2\epsilon$ for a positive constant ϵ. Then $\mathbf{d}^H(A, B) > \epsilon$.*

Proof. — Since $x \notin B^o$, an n-hemisphere H contains x and satisfies $B \cap H^o = \emptyset$ by Proposition 2.7. The proof reduces to the claim that $A \cap H$ contains a point y such that $\mathbf{d}(y, \mathbf{S}^n - H^o) > \epsilon$. Let α be the diameter of H passing through x. Let $\beta = \alpha \cap A$. The subset β is a connected segment in the convex n-ball $A \cap H$ whose endpoints are contained in $\partial(A \cap H)$. Since $\beta \ni x$ and at least one of the endpoints of β belongs to ∂A, it follows that $\mathbf{d}$-length$(\beta) > 2\epsilon$. Since we have $\beta \subset \alpha$ and $2\epsilon < \pi$, the segment β contains a point y such that

$$\epsilon < \mathbf{d}(y, \partial\alpha) \leq \pi/2.$$

As α is perpendicular to ∂H, we obtain

$$\mathbf{d}(y, \mathbf{S}^n - H^o) > \epsilon.$$

□

Proof of Proposition 2.10. — Let $x \in K^o$. Then $\mathbf{d}(x, \partial K) > 2\epsilon$ for a positive constant 2ϵ. Let N be a positive integer such that $\mathbf{d}^H(K, K_i) < \epsilon$ whenever $i > N$. By Lemma 2.11, $x \in K_i^o$ whenever $i > N$. Thus we obtain $\bigcup_{i=1}^{\infty} K_i^o \supset K^o$.

Given two convex n-balls A and B, if $\mathbf{d}^H(A, B) \leq \epsilon$ for a positive real number ϵ, then $\mathbf{d}^H(\partial A, \partial B) \leq 2\epsilon$: Suppose that $\mathbf{d}^H(\partial A, \partial B) > 2\epsilon$. Then either ∂A contains a point x such that $\mathbf{d}(x, \partial B) > 2\epsilon$ or ∂B contains a point y such that $\mathbf{d}(\partial A, y) > 2\epsilon$. It is sufficient to consider the first case: If $x \notin B$, then we have $\mathbf{d}(x, B) > 2\epsilon$ and, hence, $\mathbf{d}^H(A, B) > 2\epsilon$. If $x \in B$, then we have $x \in B^o$ and, by Lemma 2.11, $\mathbf{d}^H(A, B) > \epsilon$. Both are contradictions. □

CHAPTER 3

CONVEX SUBSETS IN THE KUIPER COMPLETIONS

In this second chapter, we begin by lifting the development pair to the real projective sphere $\mathbf{S}^n$. Then we define the holonomy cover M_h of a real projective manifold using the lifts. To make our discussion more familiar, we will define a completion, called a Kuiper completion or projective completion, by inducing the Riemannian metric of the sphere to the universal cover $\tilde{M}$ or the holonomy cover M_h and then completing them in the Cauchy sense. Then we define the ideal set to be the completion removed with $\tilde{M}$ or M_h, i.e., points infinitely far away from points of $\tilde{M}$ or M_h.

We will define convex sets in these completions, which are always "isomorphic" to ones in $\mathbf{S}^n$. Then we will introduce n-crescents, which are convex n-balls in the completions where a side or an $(n-1)$-hemisphere in the boundary lies in the ideal sets. We show how two convex subsets of the completion may intersect; their intersection properties are described by their images in $\mathbf{S}^n$ under the developing map. Finally, we describe the dipping intersection, the type of intersection which will be useful in this paper, and on which our theory of n-crescents depends heavily as we shall see in Chapter 5.

Finally, we discuss when a sequence of convex n-balls in the Kuiper complement may share a common open ball in them, the phenomenon which naturally occurs in this paper because of dipping intersection properties. When there exists a common open ball for a sequence of convex n-balls, we can find its geometric "limit" and the geometric "limits" of the sequences of their subsets in many cases. (This part is rewritten from the appendix of [**10**] but for general dimension n which creates no differences.)

Let M be a real projective n-manifold. Then M has a *development pair* $(\mathbf{dev}, h)$ of an immersion $\mathbf{dev} : \tilde{M} \to \mathbf{R}P^n$, called a developing map, and a holonomy homomorphism $h : \pi_1(M) \to \mathrm{PGL}(n+1, \mathbf{R})$ satisfying $\mathbf{dev} \circ \gamma = h(\gamma) \circ \mathbf{dev}$ for every $\gamma \in \pi_1(M)$. Such a pair is determined up to an action of an element ϑ of $\mathrm{PGL}(n+1, \mathbf{R})$ as follows:

$$(\mathbf{dev}, h(\cdot)) \mapsto (\vartheta \circ \mathbf{dev}, \vartheta \circ h(\cdot) \circ \vartheta^{-1}). \tag{3.1}$$

Developing maps are obtained by analytically extending coordinate charts in the atlas. The holonomy homomorphism is determined from the chosen developing map. (See Ratcliff [**28**] for more details.) The development pair characterizes the real projective structure, and hence another way to give a real projective structure to a manifold is to find a pair (f, k) where f is an immersion $\tilde{M} \to \mathbf{R}P^n$ which is equivariant with respect to the homomorphism k from the group of deck transformations to $\mathrm{PGL}(n+1, \mathbf{R})$.

We assume that the manifold-boundary δM of a real projective manifold M is *totally geodesic* unless stated otherwise. (Surely, M may have empty boundary) This means that for each point of δM, there exist an open neighborhood U and a lift $\phi : U \to \mathbf{S}^n$ of a chart $U \to \mathbf{R}P^n$ so that $\phi(U)$ is a nonempty intersection of a closed n-hemisphere with a simply convex open set. (By an n-hemisphere, we mean a closed hemisphere unless we mention otherwise.) δM is said to be *convex* if there exists an open neighborhood U and a chart ϕ for each point of δM so that $\phi(U)$ is a convex domain in $\mathbf{S}^n$. δM is said to be *concave* if there exists a chart (U, ϕ) for each point of δM so that $\phi(U)$ is the complement of a convex open set in an open simply convex subset of $\mathbf{S}^n$.

We remark that if M has totally geodesic boundary, then so do all of its covers. The same facts are true for convexity and concavity of boundary. Also, we will need to allow our manifold M to be a topological manifold with boundary being not smooth, especially when δM is convex or concave. This does not cause any complications as transition functions are smooth, and such manifolds can be considered as topologically imbedded submanifolds of smooth manifolds.

Lemma 3.1. — *Let M have totally geodesic boundary. Suppose that a connected totally geodesic $(n-1)$-submanifold S of M of codimension ≥ 1 intersects δM in its interior point. Then $S \subset \delta M$.*

Proof. — The intersection point must be a tangential intersection point. Since $\delta\tilde{M}$ is a closed subset of $\tilde{M}$, the set of intersection of S and $\delta\tilde{M}$ is an open and closed subset of S. Hence it must be S. □

Remark 3.2. — If δM is assumed to be convex, the conclusion holds also. This was done in [**10**] in dimension 2. The proof for the convex boundary case is the same as the dimension 2.

Remark 3.3. — Given any two real projective immersions $f_1, f_2 : N \to \mathbf{R}P^n$ on a real projective manifold N, they differ by an element of $\mathrm{PGL}(n+1, \mathbf{R})$, i.e., $f_2 = \zeta \circ f_1$ for a projective automorphism ζ as they are charts restricted to an open set, and they must satisfy the equation there, and by analyticity everywhere. Let $p : \mathbf{S}^n \to \mathbf{R}P^n$ denote the covering map. Given two real projective immersions $f_1, f_2 : N \to \mathbf{S}^n$, we have that $p \circ f_1 = \zeta \circ p \circ f_2$ for ζ in $\mathrm{PGL}(n+1, \mathbf{R})$. By equation 2.1, there exists an element ζ' of $\mathrm{Aut}(\mathbf{S}^n)$ so that $p \circ \zeta' = \zeta \circ p$ where ζ' and $A_{\mathbf{S}^n} \circ \zeta'$ are the only automorphisms satisfying the equation. This means that $p \circ f_1 = p \circ \zeta' \circ f_2$, and hence it follows easily that $f_1 = \zeta' \circ f_2$ or $f_1 = A_{\mathbf{S}^n} \circ \zeta' \circ f_2$ by analyticity of developing maps. Hence, any two real projective maps $f_1, f_2 : N \to \mathbf{S}^n$ differ by an element of $\mathrm{Aut}(\mathbf{S}^n)$.

We agree to lift our developing map **dev** to the standard sphere $\mathbf{S}^n$, the double cover of $\mathbf{R}P^n$, where we denote the lift by $\mathbf{dev}'$. Then for any deck transformation ϑ of $\tilde{M}$, we have $\mathbf{dev}' \circ \vartheta = h'(\vartheta) \circ \mathbf{dev}'$ by the above remark. Hence $\vartheta \mapsto h'(\vartheta)$ is a homomorphism, and we see easily that h' is a lift of h for the covering homomorphism $\mathrm{Aut}(\mathbf{S}^n) \to \mathrm{PGL}(n+1, \mathbf{R})$.

The pair $(\mathbf{dev}', h')$ will from now on be denoted by $(\mathbf{dev}, h)$, and they satisfy $\mathbf{dev} \circ \gamma = h(\gamma) \circ \mathbf{dev}$ for every $\gamma \in \pi_1(M)$, and moreover, given a real projective structure, $(\mathbf{dev}, h)$ is determined up to an action of ϑ of $\mathrm{Aut}(\mathbf{S}^n)$ as in equation 3.1 by the above remark.

The sphere $\mathbf{S}^n$ has the standard metric μ so that its projective structure is projectively equivalent to it; i.e., the geodesics agree. We denoted by $\mathbf{d}$ the path-metric induced from μ. From the immersion **dev**, we induce a Riemannian metric μ of $\tilde{M}$, and let $\mathbf{d}$ denote the induced path-metric on $\tilde{M}$. The Cauchy completion of $(\tilde{M}, \mathbf{d})$ is denoted by $(\check{M}, \mathbf{d})$, which we say is the *Kuiper completion* or *projective completion* of $\tilde{M}$. We define the ideal set $\tilde{M}_\infty = \check{M} - \tilde{M}$.

These sets are topologically independent of the choice of **dev** since the metrics pulled from developing maps are always quasi-isometric to one another, i.e., they differ by an element of $\mathrm{Aut}(\mathbf{S}^n)$ a quasi-isometry of $\mathbf{S}^n$ with metric $\mathbf{d}$.

Naturally, **dev** extends to a distance-decreasing map, which we denote by **dev** again. Since for each $\vartheta \in \mathrm{Aut}(\mathbf{S}^n)$, ϑ is quasi-isometric with respect to $\mathbf{d}$, and each deck transformations φ of $\tilde{M}$ locally mirror the metrical property of $h(\varphi)$, it follows that the deck transformations are quasi-isometric (see [**10**]). Thus, each deck transformation of $\tilde{M}$ extends to a self-homeomorphism of $\check{M}$. The extended map will be still called a *deck transformation* and will be denoted by the same symbol φ if so was the original deck transformation denoted. Finally, the equation $\mathbf{dev} \circ \vartheta = h(\vartheta) \circ \mathbf{dev}$ still holds for each deck transformation ϑ.

The kernel K of $h : \pi_1(M) \to \mathrm{Aut}(\mathbf{S}^n)$ is well-defined since h is well-defined up to conjugation. Since $\mathbf{dev} \circ \vartheta = \mathbf{dev}$ for $\vartheta \in K$, we see that **dev** induces a well-defined immersion $\mathbf{dev}' : \tilde{M}/K \to \mathbf{S}^n$. We say that $\tilde{M}/K$ the *holonomy* cover of M, and denote it by M_h. We identify K with $\pi_1(M_h)$. Since any real projective map $f : M_h \to \mathbf{S}^n$ equals $\vartheta \circ \mathbf{dev}'$ for ϑ in $\mathrm{Aut}(\mathbf{S}^n)$ by Remark 3.3, it follows that $\mathbf{dev} \circ \varphi$ equals $h'(\varphi) \circ \mathbf{dev}$ for each deck transformation $\varphi \in \pi_1(M)/\pi_1(M_h)$ and $h'(\varphi) \in \mathrm{Aut}(\mathbf{S}^n)$. Thus, $\varphi \mapsto h'(\varphi)$ is a homomorphism $h' : \pi_1(M)/\pi_1(M_h) \to \mathrm{Aut}(\mathbf{S}^n)$, which is easily seen to equal $h' = h \circ \Pi$ for the quotient homomorphism $\Pi : \pi_1(M) \to \pi_1(M)/\pi_1(M_h)$.

Moreover, by Remark 3.3, $(\mathbf{dev}', h')$ is determined up to an action of ϑ in $\mathrm{Aut}(\mathbf{S}^n)$ as in equation 3.1. Conversely, such a pair (f, k) where $f : M_h \to \mathbf{S}^n$ equivariant with respect to the homomorphism $k : \pi_1(M)/\pi_1(M_h) \to \mathrm{Aut}(\mathbf{S}^n)$ determines a real projective structure on M. From now on, we will denote $(\mathbf{dev}', h')$ by $(\mathbf{dev}, h)$, and call the pair a *development pair*.

Given **dev**, we may pull-back μ, and complete the path-metric $\mathbf{d}$ to obtain $\check{M}_h$, the completion of M_h, which is again called a *Kuiper or projective completion*. We define the *ideal set* $M_{h,\infty}$ to be $\check{M}_h - M_h$. As before the developing map **dev** extends to a distance-decreasing map, again denoted by **dev**, and each deck transformation extends to a self-homeomorphism $\check{M}_h \to \check{M}_h$, which we call a *deck transformation* still.

Finally, the equation $\mathbf{dev} \circ \vartheta = h(\vartheta) \circ \mathbf{dev}$ still holds for each deck transformation ϑ.

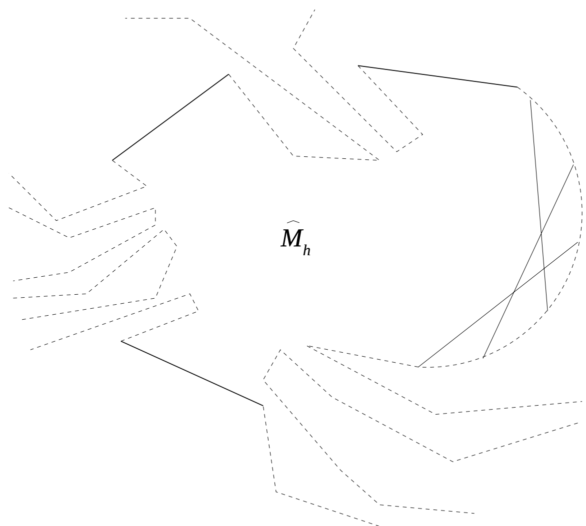

FIGURE 3.1. A figure of $\check{M}_h$. The thick dark lines indicate δM_h and the dotted lines the ideal boundary $M_{h,\infty}$, and 2-crescents in them in the right. They can have as many "pods" and what looks like "overlaps". Such pictures happen if we graft annuli into convex surfaces (see Goldman [**21**]).

As an aside, we have the following proposition. A cover M' of M is called *developing* cover if it admits a real projective immersion to $\mathbf{S}^n$. We may define the Kuiper completion $\check{M}'$ and ideal set $\tilde{M}'_\infty$ using the metric pulled back from $\mathbf{d}$ on $\mathbf{S}^n$ using the immersion. These sets are canonically defined regardless of the choice of the immersion.

Proposition 3.4. — *Let M' be a cover of M. Then M' is developing if and only if there is a covering map $g : M' \to M_h$. In this case, each real projective immersion $f : M' \to \mathbf{S}^n$ equals $\mathbf{dev} \circ g$ for a developing map $\mathbf{dev}$ of M_h.*

Proof. — The converse is clear. Let $f : M' \to \mathbf{S}^n$ be a real projective immersion. Then for any loop in M' maps to a loop in M of trivial holonomy since we may break up f to use it as charts on M' and M. Since $\pi_1(M')$ thus injects into the kernel of the holonomy homomorphism $\pi_1(M) \to \mathrm{Aut}(\mathbf{S}^n)$, M' covers M_h by a covering map g. $\mathbf{dev} \circ g$ is also a projective immersion and must agree with $k \circ f$ for $k \in \mathrm{Aut}(\mathbf{S}^n)$ by Remark 3.3. As $k^{-1} \circ \mathbf{dev} : M_h \to \mathbf{S}^n$ is a developing map, this completes the proof. □

A subset A of $\check{M}$ is a *convex segment* if $\mathbf{dev}|A$ is an imbedding onto a convex segment in $\mathbf{S}^n$. M is *convex* if given two points of the universal cover $\tilde{M}$, there exists a convex segment in $\check{M}$ connecting these two points (see Theorem 4.5). A subset A of

$\check{M}$ is *convex* if given points x and y of A, A includes a convex segment containing x and y. We say that A is a *tame* subset if it is a convex subset of $\tilde{M}$ or a convex subset of a compact convex subset of $\check{M}$. If A is tame, then $\mathbf{dev}|A$ is an imbedding onto $\mathbf{dev}(A)$ and $\mathbf{dev}|\mathrm{Cl}(A)$ for the closure $\mathrm{Cl}(A)$ of A onto a compact convex set $\mathrm{Cl}(\mathbf{dev}(A))$. The *interior* A^o of A is defined to be the set corresponding to $\mathrm{Cl}(\mathbf{dev}(A))^o$ and the boundary ∂A the subset of $\mathrm{Cl}(A)$ corresponding to $\partial\mathrm{Cl}(\mathbf{dev}(A))$. Note that ∂A may not equal the manifold boundary δA if A has a (topological) manifold structure. But if A is a compact convex set, then $\mathbf{dev}(A)$ is a manifold by Proposition 2.6, i.e., a sphere or a ball, and ∂A has to equal δA. In this case, we shall use δA over ∂A. A *side* of a compact convex subset A of $\check{M}$ is a maximal convex subset of δA. A *polyhedron* is a compact convex subset A of $\check{M}$ with finitely many sides.

Definition 3.5. — An *i-ball* A in $\check{M}$ is a compact subset of $\check{M}$ such that $\mathbf{dev}|A$ is a homeomorphism to an i-ball (not necessarily convex) in a great i-sphere and its manifold interior A^o is a subset of $\tilde{M}$. A *convex i-ball* is an i-ball that is convex.

A tame set in $\check{M}$ which is homeomorphic to an i-ball is not necessarily an i-ball in this sense; that is, its interior may not be a subset of $\tilde{M}$. We will say it is a *tame topological i-ball* but not i-ball or convex i-ball.

We define the terms convex segments, convex subset, tame subset, i-ball and convex i-ball in $\check{M}_h$ in the same manner as for $\check{M}$ and the Kuiper completions of developing covers.

We will from now on be working on $\check{M}_h$ only; *however, all of the materials in this chapter will work for $\check{M}$ and the Kuiper completions of developing covers as well, and much of the materials in the remaining chapters will work also*; *however, we will not say explicitly as the readers can easily figure out these details.*

An n-bihedron is bounded by two $(n-1)$-dimensional hemispheres; the corresponding subsets of A are the sides of A (see [**12**]).

Definition 3.6. — An n-ball A of $\check{M}_h$ is said to be an *n-bihedron* if $\mathbf{dev}|A$ is a homeomorphism onto an n-bihedron. An n-ball A of $\check{M}_h$ is said to be an *n-hemisphere* if $\mathbf{dev}|A$ is a homeomorphism onto an n-hemisphere in $\mathbf{S}^n$.

A bihedron is said to be an *n-crescent* if one of its side is a subset of $M_{h,\infty}$ and the other side is not. An n-hemisphere is said to be an *n-crescent* if a subset in the boundary corresponding to an $(n-1)$-hemisphere under $\mathbf{dev}$ is a subset of $M_{h,\infty}$ and the boundary itself is not a subset of $M_{h,\infty}$.

Note that an n-crescent in $\check{M}$ (or the Kuiper completions of developing covers) is defined in the same obvious manner.

To distinguish, a *bihedral n-crescent* is an n-crescent that is a bihedron, and a *hemispheric n-crescent* is an n-crescent that is otherwise.

In contrast to Definition 3.6, we define an *m-bihedron* for $1 \leq m \leq n-1$, to be only a tame topological m-ball whose image under $\mathbf{dev}$ is an m-bihedron in a great m-sphere in $\mathbf{S}^n$, and an *m-hemisphere*, $0 \leq m \leq n-1$, to be one whose image under $\mathbf{dev}$ is an m-hemisphere. So, we do not necessarily have $A^o \subset M_h$ when A is one of these.

Example 3.7. — Let us give two trivial examples of real projective n-manifolds to demonstrate n-crescents (see [**10**] for more 2-dimensional examples).

Let $\mathbf{R}^n$ be an affine patch of $\mathbf{S}^n$ with standard affine coordinates $x_1, x_2, \dots, x_n$ and O the origin. Consider $\mathbf{R}^n - \{O\}$ quotient out by the group $\langle g \rangle$ where $g : x \to 2x$ for $x \in \mathbf{R}^n - \{0\}$. Then the quotient is a real projective manifold diffeomorphic to $\mathbf{S}^{n-1} \times \mathbf{S}^1$. Denote the manifold by N, and we see that N_h can be identified with $\mathbf{R}^n - \{O\}$. Thus, $\check{N}_h$ equals the closure of $\mathbf{R}^n$ in $\mathbf{S}^n$; that is, $\check{N}_h$ equals an n-hemisphere H, and $N_{h,\infty}$ is the union of $\{O\}$ and the boundary great sphere $\mathbf{S}^{n-1}$ of H. Moreover, the closure of the set R given by $x_1 + x_2 + \cdots + x_n > 0$ in H is an n-bihedron and one of its side is included in $\mathbf{S}^{n-1}$. Hence, R is an n-crescent.

Let H_1 be the open half-space given by $x_1 > 0$, and l the line $x_2 = \cdots = x_n = 0$ (provided $n \geq 3$). Let g_1 be the real projective transformation given by $(x_1, x_2, \dots, x_n) \mapsto (2x_1, x_2, \dots, x_n)$ and g_2 that given by

$$(x_1, x_2, \dots, x_n) \mapsto (x_1, 2x_2, \dots, 2x_n).$$

Then the quotient manifold L of $H_1 - l$ by the commutative group generated by g_1 and g_2 is diffeomorphic to $\mathbf{S}^{n-2} \times \mathbf{S}^1 \times \mathbf{S}^1$, and we may identify its holonomy cover L_h with $H_1 - l$ and $\check{L}_h$ with the closure $\mathrm{Cl}(H_1)$ of H_1 in $\mathbf{S}^n$. Clearly, $\mathrm{Cl}(H_1)$ is an n-bihedron bounded by an $(n-1)$-hemisphere that is the closure of the hyperplane given by $x_1 = 0$ and an $(n-1)$-hemisphere in the boundary of the affine patch $\mathbf{R}^n$. Therefore, $L_{h,\infty}$ is the union of $H_1 \cap l$ and two $(n-1)$-hemispheres that form the boundary of $\mathrm{Cl}(H_1)$. $\mathrm{Cl}(H_1)$ is not an n-crescent since $\mathrm{Cl}(H_1)^o \cap L_{h,\infty} \supset l \cap H_1^o \neq \emptyset$. In fact, $\mathrm{Cl}(H_1)$ includes no n-crescents.

Let R be an n-crescent. If R is an n-bihedron, then we define α_R to be the interior of the side of R in $M_{h,\infty}$ and ν_R the other side. If R is an n-hemisphere, then we define α_R to be the union of the interiors of all $(n-1)$-hemispheres in $\delta R \cap M_{h,\infty}$ and define ν_R the complement of α_R in δR. Clearly, ν_R is a tame topological $(n-1)$-ball.

Remark 3.8. — There is another definition of n-crescents due to the referee is as follows. *An n-crescent set* in M_h is a **closed** subset R of M_h such that

- **dev** restricted to R is injective,
- $\mathbf{dev}(\mathrm{int}R)$ for the topological interior $\mathrm{int}R$ of R is an open n-hemisphere or an open n-bihedron,
- $\mathrm{bd}R \cap M_h$ is not empty,
- $\mathbf{dev}(\mathrm{bd}R \cap M_h)$ is included in an $(n-1)$-hemisphere (equivalently, $\delta\mathbf{dev}(R)$ includes an open $(n-1)$-hemisphere β_R disjoint from $\mathbf{dev}(R)$).

It is elementary to show the following: If R is an n-crescent, then $R \cap M_h$ is an n-crescent set. Conversely, the closure of an n-crescent set in $\check{M}_h$ is obviously an n-crescent. Hence, there exists a canonical correspondence between n-crescent sets and n-crescents.

Finally, this definition shows some relationship between crescents and *poche* and *coque* defined by Benzécri [**6**].

Let us now discuss about how two convex sets may meet. Let F_1 and F_2 be two convex i-, j-balls in $\check{M}_h$ respectively. We say that F_1 and F_2 overlap if $F_1^o \cap F_2^o \neq \emptyset$. This is equivalent to $F_1 \cap F_2^o \neq \emptyset$ or $F_1^o \cap F_2^o \neq \emptyset$ when F_1 and F_2 are n-balls.

Proposition 3.9. — *If F_1 and F_2 overlap, then $\mathbf{dev}|F_1 \cup F_2$ is an imbedding onto $\mathbf{dev}(F_1) \cup \mathbf{dev}(F_2)$ and $\mathbf{dev}|F_1 \cap F_2$ onto $\mathbf{dev}(F_1) \cap \mathbf{dev}(F_2)$. Moreover, if F_1 and F_2 are n-balls, then $F_1 \cup F_2$ is an n-ball, and $F_1 \cap F_2$ is a convex n-ball.*

Proof. — The proof is a direct generalization of that of Theorem 1.7 of [**10**]. We see that it follows from Proposition 3.10 since F_1 and F_2 satisfy the premise as $\mathbf{dev}(F_1)$ and $\mathbf{dev}(F_2)$ are convex. □

Proposition 3.10. — *Let A be a k-ball in $\check{M}_h$ and B an l-ball. Suppose that $A^o \cap B^o \neq \emptyset$, $\mathbf{dev}(A) \cap \mathbf{dev}(B)$ is a compact manifold in $\mathbf{S}^n$ with interior equal to $\mathbf{dev}(A^o) \cap \mathbf{dev}(B^o)$ and $\mathbf{dev}(A^o) \cap \mathbf{dev}(B^o)$ is pathwise-connected. Then $\mathbf{dev}|A \cup B$ is a homeomorphism onto $\mathbf{dev}(A) \cup \mathbf{dev}(B)$.*

Proof. — This follows as in its affine version Lemma 6 in [**16**] but is rather elementary. First, we prove injectivity: Let $x \in A$, $y \in B$, and $z \in A^o \cap B^o$ with $\mathbf{dev}(x) = \mathbf{dev}(y)$. There is a path γ in $\mathbf{dev}(A) \cap \mathbf{dev}(B)$ from $\mathbf{dev}(z)$ to $\mathbf{dev}(x)$ such that $\gamma|[0,1)$ maps into $\mathbf{dev}(A)^o \cap \mathbf{dev}(B)^o$. Since $\mathbf{dev}|A$ is an imbedding onto $\mathbf{dev}(A)$, there is a lift γ_A of γ into A^o from z to x. Similarly, there is a lift γ_B of γ into B^o from z to y. Note that $\gamma_A|[0,1)$ agrees with $\gamma_B|[0,1)$ by the uniqueness of lifts of paths for immersions. Thus, $x = y$ since $\check{M}$ is a complete metric space.

By the injectivity, there is a well-defined inverse function f to $\mathbf{dev}|A \cup B$. f restricted to $\mathbf{dev}(A)$ equals the inverse map of $\mathbf{dev}|A$, and f restricted to $\mathbf{dev}(B)$ the inverse map of $\mathbf{dev}|B$. Since both inverse maps are continuous, and $\mathbf{dev}(A)$ and $\mathbf{dev}(B)$ are closed, f is continuous. (See also [**8**].) □

In the following, we describe a useful geometric situation modeled on "dipping a bread into a bowl of milk". Let D be a convex n-ball in $\check{M}_h$ such that δD includes a tame subset α homeomorphic to an $(n-1)$-ball. We say that a convex n-ball F is *dipped into* (D, α) if the following statements hold:

- D and F overlap.
- $F \cap \alpha$ is a convex $(n-1)$-ball β with $\delta\beta \subset \delta F$ and $\beta^o \subset F^o$.
- $F - \beta$ has two convex components O_1 and O_2 such that $\mathrm{Cl}(O_1) = O_1 \cup \beta = F - O_2$ and $\mathrm{Cl}(O_2) = O_2 \cup \beta = F - O_1$.
- $F \cap D$ is equal to $\mathrm{Cl}(O_1)$ or $\mathrm{Cl}(O_2)$.

(The second item sometimes is crucial in this paper.) We say that F is *dipped into* (D, α) *nicely* if the following statements hold:

- F is dipped into (D, α).
- $F \cap D^o$ is identical with O_1 and O_2.
- $\delta(F \cap D) = \beta \cup \xi$ for a topological $(n-1)$-ball ξ, not necessarily convex or tame, in the topological boundary $\mathrm{bd}F$ of F in $\check{M}_h$ where $\beta \cap \xi = \delta\beta$.

As a consequence, we have $\delta\beta \subset \mathrm{bd}F$. (As above this is a crucial point.) (The nice dipping occurs when the bread does not touch the bowl.)

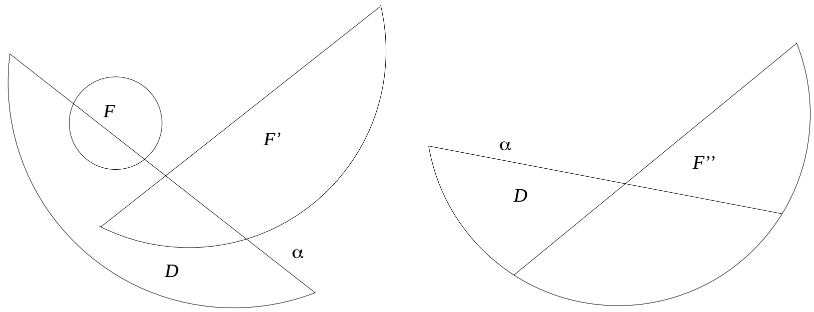

Figure 3.2. Various examples of dipping intersections. Loosely speaking α plays the role of the milk surface, F, F', and F'' the breads, and D^o the milk. The left one indicates nice dippings, and the right one not a nice one.

The direct generalization of Corollary 1.9 of [**10**] gives us:

Corollary 3.11. — *Suppose that F and D overlap, and $F^o \cap (\delta D - \alpha^o) = \emptyset$. Assume the following two equivalent conditions*:

$$\bullet \quad F^o \cap \alpha \neq \emptyset, \qquad \bullet \quad F \not\subset D.$$

Then F is dipped into (D, α). If $F \cap (\delta D - \alpha^o) = \emptyset$ furthermore, then F is dipped into (D, α) nicely. □

Example 3.12. — In Example 3.7, choose a compact convex ball B in $\mathbf{R}^n - \{O\} = N_h$ intersecting R in its interior but not included in R. Then B dips into (R, P) nicely where P is the closure of the plane given by $x_1 + \cdots + x_n = 0$. Also let S be the closure of the half plane given by $x_1 > 0$. Then S dips into (R, P) but not nicely.

Consider the closure of the set in $\check{N}_h$ given by $0 < x_1 < 1$ and that of the set $0 < x_2 < 1$. Then these two sets do not dip into each other for any choice of $(n-1)$-balls in their respective boundaries to play the role of α.

Since **dev** restricted to small open sets are charts, and the boundary of M_h is convex, each point x of M_h has a compact ball-neighborhood $B(x)$ so that $\mathbf{dev}|B(x)$ is an imbedding onto a compact convex ball in $\mathbf{S}^n$ (see Section 1.11 of [**10**]). $\mathbf{dev}(B(x))$ can be assumed to be a $\mathbf{d}$-ball with center $\mathbf{dev}(x)$ and radius $\epsilon > 0$ intersected with an n-hemisphere H so that $\delta M_h \cap B(x)$ corresponds to $\delta H \cap \mathbf{dev}(B(x))$. Of course, $\delta M_h \cap B(x)$ or $\delta H \cap \mathbf{dev}(B(x))$ may be empty. We say that such $B(x)$ is an *ϵ-tiny ball* of x and ϵ the *$\mathbf{d}$-radius* of $B(x)$.

Note that for an ϵ-tiny ball $B(x)$, $\delta M_h \cap B(x)$ is a compact convex $(n-1)$-ball or empty, and the topological boundary $\mathrm{bd}B(x)$ equals the closure of $\delta B(x)$ removed with this set.

Lemma 3.13. — *If $B(x)$ and an n-crescent R overlap, then either $B(x)$ is a subset of R or $B(x)$ is dipped into (R, ν_R) nicely.*

Proof. — Since $\mathrm{Cl}(\alpha_R) \subset M_{h,\infty}$ and $B(x) \subset M_h$, Corollary 3.11 implies the conclusion. □

As promised, we will reproduce two propositions on the sequences of convex n-balls that "converge" to a convex ball in the Kuiper completions from the Appendix of [**10**].

Recall that μ denote the Riemannian metric on $\check{M}_h$ induced from that of $\mathbf{S}^n$ by $\mathbf{dev}$.

Proposition 3.14. — *Let $\{D_i\}$ be a sequence of convex n-balls in $\check{M}_h$. Let $x \in M_h$, and $B(x)$ a tiny ball of x. Suppose that the following properties hold*:

1. *δD_i includes an $(n-1)$-ball ν_i.*
2. *$B(x)$ overlaps with D_i and does not meet $\delta D_i - \mathrm{Cl}(\nu_i)$.*
3. *A sequence $\{x_i\}$ converges to x where $x_i \in \nu_i$ for each i.*
4. *The sequence $\{\mathbf{n}_i\}$ converges where $\mathbf{n}_i$ is the outer-normal $\mathbf{d}$-unit vector to ν_i at x_i with respect to μ for each i.*

Then there exist a positive integer N and a convex open disk $\mathcal{P}$ in $B(x)$ such that

$$\mathcal{P} \subset D_i \text{ whenever } i > N.$$

Proof. — The appendix of [**10**] has the proof in dimension 2, which easily generalizes to the dimension n: We follow this proof. Let c be a positive real number so that $\mathbf{d}(x, \mathrm{bd}B(x)) > c$. Let s_i be the inward maximal segment in $D_i \cap B(x)$ with an endpoint x_i in ν_i. Choosing N_1 so that for $i > N_1$, $\mathbf{d}(x_i, x) < c/2$. Then the $\mathbf{d}$-length of s_i is greater than $c/2$ for $i > N_1$. Choose a point y_i on s_i of distance $c/4$ from x_i. Then we have $\mathbf{d}(y_i, \mathrm{bd}B(x)) > c/4$.

For each i, it is easy to see that $D_i \cap B(x)$ includes the ball $B_{c/4}(y_i)$ of $\mathbf{d}$-radius $c/2$ with center y_i. As $\mathbf{n}_i$ converges to a $\mathbf{d}$-unit vector at x, the sequence of points y_i converges to a point y of $B(x)$. Choosing N, $N > N_1$, to be so that for $i > N$ $\mathbf{d}(y_i, y) < c/8$, we obtain that $B_{c/4}(y_i) \supset B_{c/8}(y)$ for $i > N$. Letting $\mathcal{P}$ equal $B_{c/8}(y)^o$ completes the proof. □

We say that a compact subset D_∞ of $\mathbf{S}^n$ is the *resulting set* of a sequence $\{D_i\}$ of compact subsets of $\check{M}_h$ if $\{\mathbf{dev}(D_i)\}$ converges to D_∞. Let $\{D_i\}$ and $\{B_i\}$ be sequences of convex n-balls with resulting sets D_∞ and B_∞ respectively; let $\{K_i\}$ be a sequence of compact subsets with the resulting set K_∞. We say that $\{D_i\}$ *subjugates* $\{K_i\}$ if $D_i \supset K_i$ for each i and that $\{B_i\}$ *dominates* $\{D_i\}$ if B_i and D_i overlap for each i and if B_∞ includes D_∞. Moreover, we say that $\{K_i\}$ is *ideal* if there is a positive integer N for every compact subset F of M_h such that $F \cap K_i = \emptyset$ whenever $i > N$. In particular, if K_i is a subset of $M_{h,\infty}$ for each i, then $\{K_i\}$ is an ideal subjugated sequence.

Proposition 3.15. — *Suppose that $\{D_i\}$ is a sequence of n-balls including a common open ball $\mathcal{P}$, $\{B_i\}$ is another sequence of n-balls, and $\{K_i\}$ a sequence of subsets of $\check{M}_h$. Assume that $\{D_i\}$ subjugates $\{K_i\}$ and that $\{B_i\}$ dominates $\{D_i\}$. Then $\check{M}_h$ includes two convex n-balls D^u and B^u and a compact subset K^u with the following properties*:

1. *$D^u \supset \mathcal{P}$, and $\mathbf{dev}(D^u) = D_\infty$.*
2. *$B^u \supset D^u$, and $\mathbf{dev}(B^u) = B_\infty$.*
3. *$D^u \supset K^u$, and $\mathbf{dev}(K^u) = K_\infty$.*
4. *If $\{K_i\}$ is ideal, then $K^u \subset M_{h,\infty}$.*

Proof. — The proof is identical with that of Theorem 4 in the Appendix of [**10**]:

(1) Since $\mathbf{dev}(D_i)$ includes $\mathbf{dev}(\mathcal{P})$ for each i, we have $D_\infty \supset \mathbf{dev}(\mathcal{P})$; hence, D_∞ is a convex n-ball. Since Proposition 2.10 implies

$$\bigcup_{i=1}^{\infty} \mathbf{dev}(D_i)^o \supset D_\infty^o,$$

by Proposition 3.10 $\mathbf{dev}|\bigcup_{i=1}^{\infty} D_i^o$ is an imbedding onto $\bigcup_{i=1}^{\infty} \mathbf{dev}(D_i)^o$. Thus, $\bigcup_{i=1}^{\infty} D_i^o$ includes a convex disk D_{sp} such that $\mathbf{dev}|D_{sp}$ is an imbedding onto D_∞^o. If we let $D^u = \mathrm{Cl}(D_{sp})$, then (1) follows.

(2) Let $\mathcal{P}'$ be a compact convex n-ball in $\mathcal{P}^o$; let $\mathcal{P}''$ be its interior $(\mathcal{P}')^o$. Noting that $B_\infty \supset \mathbf{dev}(\mathcal{P})$, a point x of $\mathbf{dev}(\mathcal{P}'')$ satisfies $\mathbf{d}(x, \delta B_\infty) > \epsilon$ for a small positive constant ϵ. Lemma 2.11 easily shows that there is a positive integer N such that

$$\mathbf{dev}(B_i) \supset \mathbf{dev}(\mathcal{P}'') \text{ whenever } i > N.$$

Since $\mathbf{dev}|B_i \cup D_i$ is an imbedding onto $\mathbf{dev}(B_i) \cup \mathbf{dev}(D_i)$ by Proposition 3.9 and $D_i \supset \mathcal{P}$, it follows that $B_i \supset \mathcal{P}''$ whenever $i > N$. (2) follows from (1).

(3) Since $K_i \subset D_i$ for each i, we have $K_\infty \subset D_\infty$. Let $K^u = (\mathbf{dev}|D^u)^{-1}(K_\infty)$. (3) follows.

(4) We show that $K^u \subset M_{h,\infty}$. To the contrary, suppose that $K^u \cap M_h$ contains a point x. Suppose further that $x \in M_h^o$; thus, there is a tiny ball $B(x)$ satisfying $x \in B(x)^o$ and such that $\mathbf{dev}(B(x))^o \cap D_\infty^o$ is star-shaped from a point y of $\mathbf{dev}(\mathcal{P})$. (A *star-shaped* subset of $\mathbf{S}^n$ from a point is a subset such that each of its elements can be connected by a simply convex segment in it from the point.) We obtain by Proposition 3.10 that

$$\mathbf{dev}|D_i \cup B(x) \cup D^u$$

is an imbedding onto

$$\mathbf{dev}(D_i) \cup \mathbf{dev}(B(x)) \cup D_\infty$$

for each i.

Since $\{\mathbf{dev}(K_i)\}$ converges to K_∞ and $\mathbf{dev}(B(x))^o$ is an open neighborhood of $\mathbf{dev}(x)$, there is a positive integer N such that

$$\mathbf{dev}(K_i) \cap \mathbf{dev}(B(x))^o \neq \emptyset \text{ whenever } i > N.$$

Let i be an integer greater than N. The open disk $B(x)^o$ includes a non-empty subset δ_i defined by

$$\delta_i = (\mathbf{dev}|B(x)^o)^{-1}(\mathbf{dev}(K_i) \cap \mathbf{dev}(B(x))^o).$$

By the conclusion of the second paragraph above, δ_i is a subset of K_i. Since we have $\delta_i \subset B(x)$ whenever $i > N$, this contradicts the premise on $\{K_i\}$.

Finally, suppose that $x \in \delta M_h$. Let us extend M_h by attaching a small open n-ball in $\mathbf{S}^n$ around x by a projective map. The resulting projective surface still has convex boundary. Now the previous argument applies and yields a contradiction again. ☐

As an immediate application, we have

Corollary 3.16. — *Suppose that M is not projectively diffeomorphic to an open n-bihedron or n-hemisphere. Let R_i be a sequence of n-crescents such that a sequence of points $x_i \in \nu_{R_i}$ converges to a point x of M_h. Then M_h includes a crescent R containing x so that $\mathbf{dev}(R_{i_j})$ converges to $\mathbf{dev}(R)$ for a subsequence R_{i_j} of R_i and R and R_{i_j} include a common open ball $\mathcal{P}$ for j sufficiently large. Finally if R_i are n-hemispheres, then so is R. If R_i are n-bihedrons, then R is either an n-hemisphere or an n-bihedron.*

Proof. — Assume without loss of generality that $x_i \in B(x)$ and that the sequence of $\mathbf{d}$-unit outer normal vectors n_i at x_i converges to one at x. Then by Proposition 3.14, there exists a common ball $\mathcal{P}$ in $R_i \cap B(x)$ for i sufficiently large.

Note that $\mathrm{Cl}(\alpha_i) \subset R_i$. By choosing a subsequence, we may assume that $\mathbf{dev}(\mathrm{Cl}(\alpha_i))$ converges to a compact set α so that $\mathrm{Cl}(\alpha_i)$ forms a subjugated sequence of R_i. Moreover $\mathrm{Cl}(\alpha_i)$ forms an ideal one since $\mathrm{Cl}(\alpha_i)$ never meets any compact subset of M_h.

By Proposition 3.15, there exists a convex n-ball R in $\check{M}_h$ so that a subsequence of $\mathbf{dev}(R_i)$ converges to $\mathbf{dev}(R)$. Since each $\mathrm{Cl}(\alpha_i)$ includes an $(n-1)$-hemisphere, so does α by Proposition 2.8. Hence, $\mathbf{dev}(R)$ is an n-hemisphere or an n-bihedron by Lemma 2.5. Furthermore, a subset α^u of R which maps to α includes an $(n-1)$-hemisphere. If δR is a subset of $M_{h,\infty}$, then M_h is diffeomorphic to an open n-bihedron or n-hemisphere by the following lemma 3.17. Thus, R is an n-crescent as α^u is a subset of $M_{h,\infty}$ by Proposition 3.15.

Since R includes $\mathcal{P}$, R overlaps $B(x)$. Hence, $\mathbf{dev}|R \cup B(x)$ is a homeomorphism onto $\mathbf{dev}(R) \cup \mathbf{dev}(B(x))$. Since $\mathbf{dev}(R)$ contains $\mathbf{dev}(x)$, we have that $x \in R$.

The two last statements follow from Proposition 2.8. $\square$

Lemma 3.17. — *Suppose that $\check{M}_h$ includes an n-ball B with $\delta B \subset M_{h,\infty}$. Then M_h equals B^o.*

Remark 3.18. — We can relax the condition $x_i \in \nu_{R_i}$ to $x_i \in R_i$ in Corollary 3.16. The proof requires us to choose a smaller crescent S_i in R_i so that $x_i \in \nu_{S_i}$ and $\alpha_{S_i} \subset \alpha_{R_i}$. The rest of the straightforward proof is left to the reader.

PART II

$(n-1)$-CONVEXITY AND DECOMPOSITION

CHAPTER 4

$(n-1)$-CONVEXITY AND n-CRESCENTS

In this chapter, we introduce m-convexity. Then we state Theorem 4.6 central to this chapter, which relates the failure of $(n-1)$-convexity with an existence of n-crescents, or half-spaces. The proof of theorem is similar to what is in Section 5 in [**10**]. Let M be a real projective n-manifold with empty or totally geodesic boundary; let M_h be the holonomy cover and $\check{M}_h$ the Kuiper completion of M_h. An *m-simplex* T in $\check{M}_h$ is a tame subset of $\check{M}_h$ such that $\mathbf{dev}|T$ is an imbedding onto an affine m-simplex in an affine patch in $\mathbf{S}^n$. If M is compact but not $(n-1)$-convex, then we can show that there exists an n-simplex T in $\check{M}_h$ with sides $F_1, \dots, F_{n+1}$ so that $T \cap M_{h,\infty}$ is a nonempty subset of the interior of F_1. We first choose a sequence of points q_i of F_1 converging to a point x in $F_1 \cap M_{h,\infty}$. Then we pull back q_i to points p_i in the closure of a fundamental domain by a deck transformation ϑ_i^{-1}. Then analogously to [**10**], we show that $T_i = \vartheta_i^{-1}(T)$ "converges to" a nondegenerate convex n-ball. Showing that $\mathbf{dev}(T_i)$ converges to an n-bihedron or an n-hemisphere is more complicated than in [**10**]. The idea of the proof is to show that the sequence of the images under ϑ_i of the ϵ-$(n-1)$-$\mathbf{d}$-balls in $\vartheta^{-1}(F_1)$ with center p_i often have to degenerate to a point when x is chosen specially. So when pulled back by ϑ_i^{-1}, the balls become standard ones again, and F_1 must blow up to be an $(n-1)$-hemisphere under ϑ_i^{-1}.

Definition 4.1. — We say that M is *m-convex*, $0 < m < n$, if the following holds. If $T \subset \check{M}_h$ is an $(m+1)$-simplex with sides $F_1, F_2, \dots, F_{m+2}$ such that $T^o \cup F_2 \cup \dots \cup F_{m+2}$ does not meet $M_{h,\infty}$, then T is a subset of M_h.

Proposition 4.2. — *Let T be an affine $(m+1)$-simplex in an affine space with sides $F_1, F_2, \dots, F_{m+2}$. The following are equivalent*:

(a) *M is m-convex.*
(b) *Any real projective immersion f from $T^o \cup F_2 \cup \dots \cup F_{m+2}$ to M extends to one from T.*
(c) *Every cover of M is m-convex.*

Proof. — The proof of the equivalence of (a) and (b) is the same as the affine version Lemma 1 in [**16**]: Suppose that M is not m-convex, and let $p : \tilde{M} \to M$ denote the universal covering map. Then there exists a d-simplex T in $\check{M}$ such that $T \cap \tilde{M}_\infty = F_1^o \cap \tilde{M}_\infty \neq \emptyset$ for a side F_1 of T. Since T is a d-simplex, $\mathbf{dev}|T$ is an imbedding onto $\mathbf{dev}(T)$, and $\mathbf{dev}(T)$ is a d-simplex in an affine patch of $\mathbf{R}P^n$. Let f be the map $(\mathbf{dev}|\mathbf{dev}(T))^{-1}$ restricted to $\mathbf{dev}(T^o) \cup \mathbf{dev}(F_2) \cup \cdots \cup \mathbf{dev}(F_{m+2})$. f is a projective immersion to $\tilde{M}$. It is easy to see that f does not extend to all of $\mathbf{dev}(T)$. Hence, $p \circ f$ is an affine immersion which does not extend to $\mathbf{dev}(T)$.

Suppose that M is m-convex. Let $\mathbf{dev} : \check{M} \to \mathbf{R}P^n$ be the developing map. Let $f : T^o \cup F_2 \cup \cdots \cup F_{m+2}$ be a projective immersion into M, and let $\tilde{f}$ be the lift of f to $\tilde{M}$. Then $\mathbf{dev} \circ \tilde{f}$ is also a real projective immersion from $T^o \cup F_2 \cup \cdots \cup F_{m+2}$ into $\mathbf{R}P^n$. Since the rank of the map $\mathbf{dev} \circ \tilde{f}$ is maximal, the map extends to a global real projective transformation ϕ on $\mathbf{R}P^n$, and hence $\mathbf{dev} \circ \tilde{f}$ is an imbedding. Since $\mathbf{dev}|\tilde{f}(T^o)$ is an imbedding onto an open m-simplex $\mathbf{dev} \circ \tilde{f}(T^o)$ in $\mathbf{R}P^n$, and $\tilde{f}(T^o)$ is a convex subset of $\tilde{M}$, it follows that the closure T' of $\tilde{f}(T^o)$ is a tame subset of $\check{M}$ so that $\mathbf{dev}|T'$ is an imbedding onto $\mathrm{Cl}(\tilde{f}(T^o))$.

Since T' is an $(m+1)$-simplex with sides $\tilde{f}(F_2), \ldots, \tilde{f}(F_{m+2})$, and a remaining side F_1'. Since T^o and $\tilde{f}(F_2), \ldots \tilde{f}(F_{m+2})$ are subsets of $\tilde{M}$, and M is m-convex, we have $T' \subset \tilde{M}$. Therefore the affine embedding $f' : T \to T'$ given as $(\mathbf{dev}|T')^{-1} \circ \phi$ extends $\tilde{f}$ and $p \circ f'$ extends f.

The equivalence of (b) and (c) follows from the fact that a real projective map to M always lifts to its cover. □

Proposition 4.3. — *M is not m-convex if and only if there exists an $(m+1)$-simplex with a side F_1 such that $T \cap M_{h,\infty} = F_1^o \cap M_{h,\infty} \neq \emptyset$.*

Proof. — This elementary proof is same as Lemma 3 in [**16**]. Suppose that every $(m+1)$-simplex T has the property that T do not meet $M_{h,\infty}$ or $T \cap M_{h,\infty}$ is a subset of the union of two or more sides but not less than two sides or if $T \cap \tilde{M}_\infty$ is a subset of a side F, then $F \cap \tilde{M}_\infty$ is not a subset of F^o. Then one sees easily that the definition for m-convexity is satisfied by M.

Conversely, if M is m-convex, and T is an $(m+1)$-simplex with sides $F_1, \ldots, F_{m+2}$ such that $T^o \cup F_2 \cup \cdots \cup F_{m+2} \subset M_h$, then $T \subset \tilde{M}$. Consequently, there is no $(m+1)$-simplex T with $T \cap M_{h,\infty} = F_1^o \cap M_{h,\infty} \neq \emptyset$. □

Remark 4.4. — It is easy to see that i-convexity implies j-convexity whenever $i \leq j < n$. (See Remark 2 in [**16**]. The proof is the same.)

Theorem 4.5. — *The following are equivalent*: *M is* 1*-convex*; *M is convex*; *M is real projectively isomorphic to a quotient of a convex domain in $\mathbf{S}^n$.*

The proof is similar to Lemma 8 in [**16**]. Since there are minor differences between affine and real projective manifolds, we will prove this theorem in Appendix A.

Let us give examples of real projective n-manifolds one of which is not $(n-1)$-convex and the other $(n-1)$-convex.

As in Figure 4.1, $\mathbf{R}^3$ removed with a complete affine line or a closed wedge, i.e. a set defined by the intersection of two half-spaces with non-parallel boundary planes is obviously 2-convex. But $\mathbf{R}^3$ removed with a discrete set of points or a convex

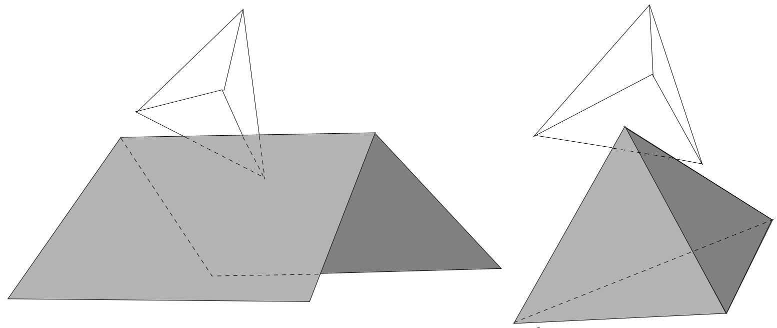

FIGURE 4.1. The tetrahedron in the left fails to detect non-2-convexity but the right one is detecting non-2-convexity.

cone defined as the intersection of three half-spaces with boundary planes in general position is not 2-convex.

We recall Example 3.7. Let $\mathbf{R}^n$ be an affine patch of $\mathbf{S}^n$ with standard affine coordinates $x_1, x_2, \dots, x_n$ and O the origin. Consider $\mathbf{R}^n - \{O\}$ quotient out by the group $\langle g \rangle$ where $g : x \to 2x$ for $x \in \mathbf{R}^n - \{0\}$. Then the quotient is a real projective manifold diffeomorphic to $\mathbf{S}^{n-1} \times \mathbf{S}^1$. We denoted the manifold by N, and we see that N_h can be identified with $\mathbf{R}^n - \{0\}$. Thus, $\check{N}_h$ equals the closure of $\mathbf{R}^n$ in $\mathbf{S}^n$; that is, $\check{N}_h$ equals an n-hemisphere H, and $N_{h,\infty}$ is the union of $\{O\}$ and the boundary great sphere $\mathbf{S}^{n-1}$ of H. Consider an n-simplex T in $\mathbf{R}^n$ given by $x_i \leq 1$ for every i and $x_1 + x_2 + \cdots + x_n \geq 0$. Then the side of T corresponding to $x_1 + x_2 + \cdots + x_n = 0$ contains the ideal point O in its interior. Therefore, N is not $(n-1)$-convex. Moreover, the closure of the set given by $x_1 + x_2 + \cdots + x_n > 0$ in $\check{M}_h = H$ is an n-bihedron and one of its side is included in $\mathbf{S}^{n-1}$. Hence, it is an n-crescent. (It will aid understanding to apply each course of the proof of Theorem 4.6 to this example.)

Let H_1 be the open half-space given by $x_1 > 0$, and l the line $x_2 = \cdots = x_n = 0$. Let g_1 be the real projective transformation given by $(x_1, x_2, \dots, x_n) \mapsto (2x_1, x_2, \dots, x_n)$ and g_2 that given by $(x_1, x_2, \dots, x_n) \mapsto (x_1, 2x_2, \dots, 2x_n)$. Then the quotient manifold L of $H_1 - l$ by the commutative group generated by g_1 and g_2 is diffeomorphic to $\mathbf{S}^{n-2} \times \mathbf{S}^1 \times \mathbf{S}^1$, and we may identify L_h with $H_1 - l$ and $\check{L}_h$ with the closure $\mathrm{Cl}(H_1)$ of H_1 in $\mathbf{S}^n$. Clearly, $\mathrm{Cl}(H_1)$ is an n-bihedron bounded by an $(n-1)$-hemisphere that is the closure of the hyperplane given by $x_1 = 0$ and an $(n-1)$-hemisphere in the boundary of the affine patch $\mathbf{R}^n$. Therefore, $L_{h,\infty}$ is the union of $H_1 \cap l$ and two $(n-1)$-hemispheres that form the boundary of $\mathrm{Cl}(H_1)$. The intersection of an n-simplex T in $\mathbf{S}^n$ with the boundary $(n-1)$-hemispheres or l is not a subset of the interior of a side of T. It follows from this that L is $(n-1)$-convex.

The main purpose of this chapter is to prove the following principal theorem:

Theorem 4.6. — *Suppose that a compact real projective manifold M with empty or totally geodesic boundary is not $(n-1)$-convex. Then the completion $\check{M}_h$ of the holonomy cover M_h includes an n-crescent.*

We may actually replace the word "totally geodesic boundary" with "convex boundary" and the proof is same step by step. However, we need this result at only one point of the paper so we do not state it.

We can also show that the completion $\check{M}$ of the universal cover $\tilde{M}$ also includes an n-crescent. The proof is identical with $\check{M}$ replacing $\check{M}_h$. Another way to do this is of course as follows: once we obtain an n-crescent in $\check{M}_h$ we may lift it to one in $\check{M}$ (see Proposition 8.13).

Remark 4.7. — As M is not $(n-1)$-convex, we may assume that $\tilde{M}$ or M_h is not projectively diffeomorphic to an open n-bihedron or an open n-hemisphere: If otherwise, $\tilde{M}$ is convex and hence $(n-1)$-convex. We will need this weaker but important hypothesis later.

A point x of a convex subset A of $\mathbf{S}^n$ is said to be *exposed* if there exists a supporting great $(n-1)$-sphere H at x such that $H \cap A = \{x\}$ (see Chapter 2 and Berger [**7**, p. 361]).

To prove Theorem 4.6, we follow Section 5 of [**10**]: Since M is not $(n-1)$-convex, $\check{M}_h$ includes an n-simplex T with a side F_1 such that $T \cap M_{h,\infty} = F_1^o \cap M_{h,\infty} \neq \emptyset$ by Proposition 4.3, where $\mathbf{dev}|T : T \to \mathbf{dev}(T)$ is an imbedding onto the n-simplex $\mathbf{dev}(T)$. Let K be the convex hull of $\mathbf{dev}(F_1 \cap M_{h,\infty})$ in $\mathbf{dev}(F_1)^o$, which is simply convex as $\mathbf{dev}(F_1)$ is simply convex.

As K is simply convex, we see that K can be considered as a bounded convex subset of an affine patch, i.e., an open n-hemisphere. We see easily that K has an exposed point in the affine sense in the open hemisphere, which is easily seen to be an exposed point in our sense as a hyperplane in the affine patch is the intersection of a hypersphere with the affine patch.

Let x' be an exposed point of K. Then $x' \in \mathbf{dev}(F_1 \cap M_{h,\infty})$, and there exists a line s' in the complement of K in $\mathbf{dev}(F_1)^o$ ending at x'. Let x and s be the inverse images of x' and s' in F_1^o respectively.

Let F_i for $i = 2, \dots, n+1$ denote the sides of T other than F_1. Let v_i for each i, $i = 1, \dots, n+1$, denote the vertex of T opposite to F_i. Let us choose a monotone sequence of points q_i on s converging to x with respect to $\mathbf{d}$.

Choose a fundamental domain F in M_h such that for every point t of F, there exists a 2ϵ-tiny ball of t in M_h for a positive constant ϵ independent of t. We assume $\epsilon \leq \pi/8$ for convenience. Let us denote by $F_{2\epsilon}$ the closure of the 2ϵ-$\mathbf{d}$-neighborhood of F, and F_ϵ that of the ϵ-$\mathbf{d}$-neighborhood of F so that F_ϵ and $F_{2\epsilon}$ are compact subsets of M_h.

For each natural number i, we choose a deck transformation ϑ_i and a point p_i of F so that $\vartheta_i(p_i) = q_i$. We let $v_{j,i}, F_{j,i}$, and T_i, $i = 1, 2, \dots$, $j = 1, \dots, n+1$, denote the images under ϑ_i^{-1} of v_j, F_j, and T respectively. Let n_i denote the outer-normal vector to $F_{1,i}$ at p_i with respect to the spherical Riemannian metric μ of M_h.

We choose subsequences so that each sequence consisting of

$$\mathbf{dev}(v_{j,i}), \mathbf{dev}(F_{j,i}), \mathbf{dev}(T_i), n_i, \text{ and } p_i$$

converge geometrically with respect to $\mathbf{d}$ for each j, $j = 1, \dots, n+1$ respectively. Since p_i belongs to the fundamental domain F for each i, the limit p of the sequence of p_i belongs to $\mathrm{Cl}(F)$. We choose an ϵ-tiny ball $B(p)$ of p. We may assume without loss of generality that p_i belongs to the interior $\mathrm{int}B(p)$ of $B(p)$. Since the action of the deck transformation group is properly discontinuous on M_h and $F_{j,i} = \vartheta_i^{-1}(F_j)$ for a compact subset F_j of M_h, there exists a natural number N such that

$$F_{2\epsilon} \cap F_{j,i} = \emptyset \text{ for each } j, i, j > 1, i > N; \tag{4.1}$$

so $B(p) \cap F_{j,i} = \emptyset$ for $j > 1$. (This corresponds to Lemma 5.4 in [**10**].) Hence, $B(p) \subset T_i$ or $B(p)$ dips into $(T_i, F_{1,i})$ for each i, $i > N$, by Corollary 3.11.

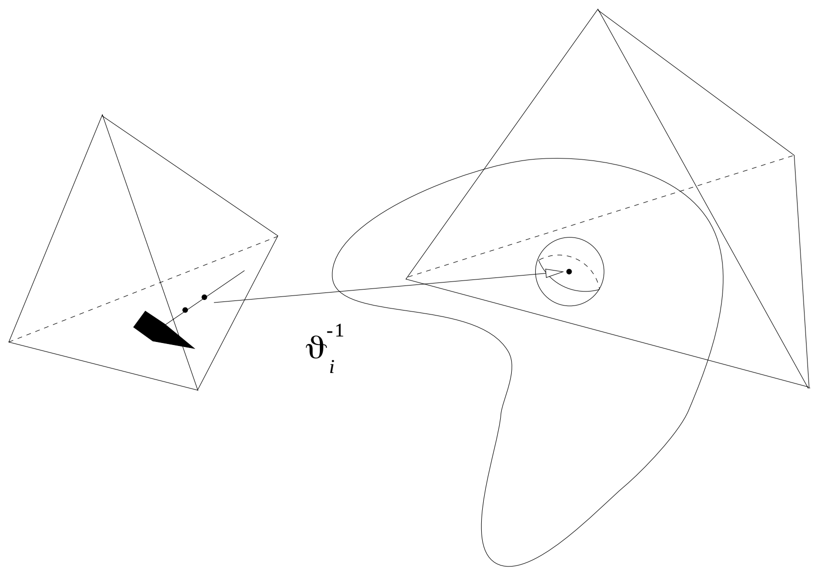

FIGURE 4.2. The pull-back process

By Proposition 3.14, there exists an integer N_1, $N_1 > N$, such that T_i includes a common open ball for $i > N_1$. Let T_∞ be the limit of $\mathbf{dev}(T_i)$. Since $\mathbf{dev}(T_i)$ includes a common ball for $i > N_1$, Proposition 2.8 shows that T_∞ is a closed convex n-ball in $\mathbf{S}^n$.

Let $F_{j,\infty}$ denote the limit of $\mathbf{dev}(F_{j,i})$. Then $\bigcup_{j=1}^{n+1} F_{j,\infty}$ is the boundary δT_∞ by Proposition 2.10.

Proposition 3.15 implies that $\check{M}_h$ includes a convex n-ball T^u and convex sets F_j^u such that $\mathbf{dev}$ restricted to them are imbeddings onto T_∞ and $F_{j,\infty}$ respectively. We have $F_j^u \subset M_{h,\infty}$ for $j \geq 2$ from the same proposition since $F_{j,i}$ is ideal.

As we shall prove below that $F_{1,\infty}$ is an $(n-1)$-hemisphere, T_∞ is a compact convex n-ball in $\mathbf{S}^n$ including the $(n-1)$-hemisphere $F_{1,\infty}$ in its boundary δT_∞. By Lemma 2.5, T_∞ is an n-bihedron or an n-hemisphere. As $\bigcup_{j\geq 2} F_j^u$ is a subset of $M_{h,\infty}$, if $F_1^u \subset M_{h,\infty}$, then $\check{M}_h = T^u$ and M_h equals the interior of T^u by Lemma 3.17. M_h is not projectively diffeomorphic to an open n-bihedron or an open n-hemisphere (see Remark 4.7); F_1^u is not a subset of $M_{h,\infty}$. Since T^u is bounded by F_1^u and

$$F_2^u \cup \cdots \cup F_{n+1}^u \subset M_{h,\infty},$$

it follows that T is an n-crescent. This completes the proof of Theorem 4.6. $\square$

We will now show that F_1^u is an $(n-1)$-dimensional hemisphere. This corresponds to Lemma 5.5 of [**10**] showing that one of the sides is a segment of $\mathbf{d}$-length π. (The following process may require us to choose further subsequences of T_i. However, since $\mathbf{dev}(F_{1,i})$ is assumed to converge to $F_{1,\infty}$, we see that we need to only show that a *subsequence* of $\mathbf{dev}(F_{1,i})$ converges to an $(n-1)$-hemisphere.)

The sequence $\mathbf{dev}(q_i) = h(\vartheta_i)\mathbf{dev}(p_i)$ converges to x'. Since p_i belongs to F, M_h includes an ϵ-tiny ball $B(p_i)$ and a 2ϵ-tiny ball $B'(p_i)$ of p_i. Let $W(p_i) = F_{1,i} \cap B(p_i)$ and $W'(p_i) = F_{1,i} \cap B'(p_i)$. We assume that $i > N_1$ from now on.

We now show that $W(p_i)$ and $W(p_i')$ are "whole" $(n-1)$-balls of $\mathbf{d}$-radius ϵ and 2ϵ, i.e., they map to such balls in $\mathbf{S}^n$ under $\mathbf{dev}$ respectively, or they are not "cut off" by the boundary $\delta F_{1,i}$:

If $p_i \in \delta M_h$, then the component L of $F_{1,i} \cap M_h$ containing p_i is a subset of δM_h by Lemma 3.1. This component is a submanifold of δM_h with boundary $\delta F_{1,i}$. Since $\delta F_{1,i}$ is a subset of $\bigcup_{j\geq 2} F_{j,i}$, and $B(p_i)$ is disjoint from it by equation 4.1, $\delta M_h \cap B(p_i)$ is a subset of L^o. Thus, $W(p_i)$ equals the convex $(n-1)$-ball $\delta M_h \cap B(p_i)$ with boundary in $\mathrm{bd} B(p_i)$ and is a $\mathbf{d}$-ball in $F_{1,i}^o$ of dimension $(n-1)$ of $\mathbf{d}$-radius ϵ and center p_i, and certainly maps to an $(n-1)$-ball of $\mathbf{d}$-radius ϵ with center $\mathbf{dev}(p_i)$.

If $p_i \in M_h^o$, then since $F_{1,i}$ passes through p_i, and $F_{j,i} \cap B(p_i) = \emptyset$ for $j \geq 2$, it follows that $B(p_i)$ dips into $(T_i, F_{1,i})$ nicely by Corollary 3.11. Thus $W(p_i)$ is an $(n-1)$-ball with boundary in $\mathrm{bd} B(p_i)$, and an ϵ-$\mathbf{d}$-ball in $F_{1,i}^o$ of dimension $(n-1)$ with center p_i.

Similar reasoning shows that $W'(p_i)$ is a 2ϵ-$\mathbf{d}$-ball in $F_{1,i}^o$ of dimension $(n-1)$ with center p_i for each i.

Since $\vartheta_i(W(p_i)) \subset F_1$, and $\mathbf{dev}(F_1)$ is a compact set, we may assume without loss of generality by choosing subsequences of ϑ_i that the sequence of the subsets $\mathbf{dev}(\vartheta_i(W(p_i)))$ of $\mathbf{dev}(F_1)$, equal to $h(\vartheta_i)(\mathbf{dev}(W(p_i)))$, converges to a set W_∞ containing x' in $\mathbf{dev}(F_1)$. Since $\mathbf{dev}|T^u$ is an imbedding onto T_∞, there exists a compact tame subset W^u in F_1 such that $\mathbf{dev}$ restricted to W^u is an imbedding onto W_∞. $\vartheta_i(W(p_i))$ is a subjugated sequence of the sequence of convex n-balls that equal T always. Since $W(p_i)$ is a subset of a compact set F_ϵ, it follows that $\vartheta_i(W(p_i))$ is ideal, and $W^u \subset M_{h,\infty}$ by Proposition 3.15. We obtain $W^u \subset F_1 \cap M_{h,\infty}$.

For the proof of the next proposition, the fact that x' is exposed will play a role:

Proposition 4.8. — *W_∞ consists of the single point x'.*

Suppose not. Then as $\mathbf{dev}(\vartheta_i(W(p_i))$ does not converge to a point, there has to be a sequence $\{\mathbf{dev}(\vartheta_i(z_i))\}$, $z_i \in W(p_i)$, converging to a point z' distinct from the limit x' of $\{\mathbf{dev}(q_i)\}$. Since we have $\mathbf{dev}(q_i) = \mathbf{dev}(\vartheta_i(p_i))$, we choose s_i to be the $\mathbf{d}$-diameter of $W(p_i)$ containing z_i and p_i, as a center. We obtained a sequence of segments $s_i \in W(p_i)$ passing through p_i of $\mathbf{d}$-length 2ϵ so that the sequence of segments $\mathbf{dev}(\vartheta_i(s_i))$ in $\mathbf{dev}(F_1)$ converges to a nontrivial segment s containing x', and z', satisfying $s \subset W_\infty \subset \mathbf{dev}(M_{h,\infty} \cap F_1)$.

Since s is a nontrivial segment, the $\mathbf{d}$-length of $h(\vartheta_i)(\mathbf{dev}(s_i))$ is bounded below by a positive constant δ independent of i. Since $h(\vartheta_i)(\mathbf{dev}(s_i))$ is a subset of the

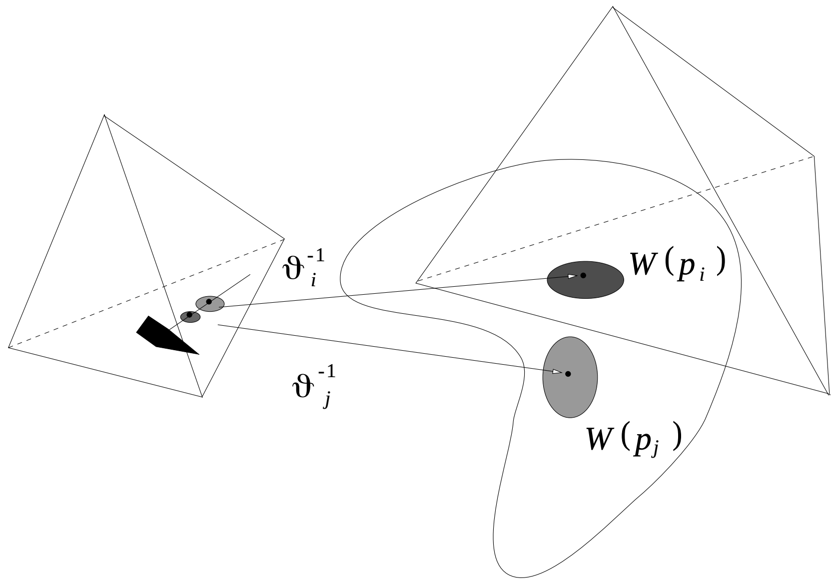

FIGURE 4.3. The pull-back process with $W(p_i)$.

$(n-1)$-simplex $\mathbf{dev}(F_1)$, which is a simply convex compact set, the $\mathbf{d}$-length of $h(\vartheta_i)(\mathbf{dev}(s_i))$ is bounded above by $\pi - \delta'$ for some small positive constant δ'. Let s_i' be the maximal segment in $W'(p_i)$ including s_i. Then the $\mathbf{d}$-length of $h(\vartheta_i)(\mathbf{dev}(s_i'))$ also belongs to the interval $[\delta, \pi - \delta']$.

Lemma 4.9. — *Let $\mathbf{S}^1$ be a great circle and o, s, p, q distinct points on a segment I in $\mathbf{S}^1$ of $\mathbf{d}$-length $< \pi$ with endpoints o, s and p between o and q. Let f_i be a sequence of projective maps $I \to \mathbf{S}^n$ so that $\mathbf{d}(f_i(o), f_i(s))$ and $\mathbf{d}(f_i(p), f_i(q))$ lie in the interval $[\eta, \pi-\eta]$ for some positive constant η independent of i. Then all of the $\mathbf{d}$-distances between $f_i(o), f_i(s), f_i(p)$, and $f_i(q)$ are bounded below by a positive constant independent of i.*

Proof. — Recall the well-known formula for cross-ratios (see [**7**]):

$$[f_i(o), f_i(s); f_i(q), f_i(p)] = \frac{\sin(\mathbf{d}(f_i(o), f_i(q)))}{\sin(\mathbf{d}(f_i(s), f_i(q)))} \frac{\sin(\mathbf{d}(f_i(s), f_i(p)))}{\sin(\mathbf{d}(f_i(o), f_i(p)))}.$$

Suppose that $\mathbf{d}(f_i(o), f_i(p)) \to 0$. Then since

$$\begin{aligned} \mathbf{d}(f_i(o), f_i(q)) &= \mathbf{d}(f_i(o), f_i(p)) + \mathbf{d}(f_i(p), f_i(q)) \geq \eta, \\ \mathbf{d}(f_i(o), f_i(q)) &\leq \mathbf{d}(f_i(o), f_i(s)) \leq \pi - \eta, \end{aligned}$$

it follows that $\sin(\mathbf{d}(f_i(o), f_i(q)))$ is bounded below and above by $\sin(\eta)$ and 1 respectively. Similarly, so is $\sin(\mathbf{d}(f_i(s), f_i(p)))$. Therefore, the right side of the equation goes to $+\infty$, while the left side remains constant since f_i is projective. This is a contradiction, and $\mathbf{d}(f_i(o), f_i(p))$ is bounded below by a positive constant.

Similarly, we can show that $\mathbf{d}(f_i(s), f_i(q))$ is bounded below by a positive constant. The conclusion follows from these two statements. □

Let $\mathbf{S}^1$ be the unit circle in the plane $\mathbf{R}^2$. Let θ, $-\pi/2 < \theta < \pi/2$, denote the point of $\mathbf{S}^1$ corresponding to the unit vector having an oriented angle of θ with $(1,0)$ in $\mathbf{R}^2$. Since s_i and s'_i are the diameters of balls of $\mathbf{d}$-radius ϵ and 2ϵ with center p_i respectively, for the segment $[-2\epsilon, 2\epsilon]$ consisting of points θ satisfying $-2\epsilon \leq \theta \leq 2\epsilon$ in $\mathbf{S}^1$, we parameterize s'_i by a projective map $f_i : [-2\epsilon, 2\epsilon] \to s'_i$, isometric with respect to $\mathbf{d}$, so that the endpoints of s'_i correspond to -2ϵ and 2ϵ, the endpoints of s_i to $-\epsilon$ and ϵ, and p_i to 0.

Lemma 4.9 applied to $k_i = h(\vartheta_i) \circ \mathbf{dev} \circ f_i$ shows that

$$\mathbf{d}(k_i(2\epsilon), k_i(\epsilon)) \text{ and } \mathbf{d}(k_i(-2\epsilon), k_i(-\epsilon))$$

are bounded below by a positive constant since $\mathbf{d}(k_i(\epsilon), k_i(-\epsilon))$ and $\mathbf{d}(k_i(2\epsilon), k_i(-2\epsilon))$ are bounded below by a positive number δ and above by $\pi - \delta'$. Since $k_i(2\epsilon)$ and $k_i(-2\epsilon)$ are endpoints of $h(\vartheta_i)(\mathbf{dev}(s'_i))$ and $k_i(\epsilon)$ and $k_i(-\epsilon)$ those of $h(\vartheta_i)(\mathbf{dev}(s_i))$, a subsequence of $h(\vartheta_i)(\mathbf{dev}(s'_i))$ converges to a segment s' in $\mathbf{dev}(F_1)$ including s in its interior. Hence, s' contains x' in its interior.

Since s'_i is a subset of $F_{2\epsilon}$, a compact subset of M_h, it follows that the corresponding subsequence of $\vartheta_i(s'_i)$ is ideal in F_1. Hence $s' \subset \mathbf{dev}(F_1 \cap M_{h,\infty}) \subset K$ by Proposition 3.15. Since x' is not an endpoint of s' but an interior point, this contradicts our earlier choice of x' as an exposed point of K. □

Since W_∞ consists of a point, it follows that the sequence of the $\mathbf{d}$-diameter of $h(\vartheta_i)(\mathbf{dev}(W(p_i)))$ converges to zero, and the sequence converges to the singleton $\{x'\}$.

Let us introduce a $\mathbf{d}$-isometry g_i, which is a real projective automorphism of $\mathbf{S}^n$, for each i so that each $g_i(\mathbf{dev}(W(p_i)))$ is a subset of the great sphere $\mathbf{S}^{n-1}$ including $\mathbf{dev}(F_1)$, and hence $h(\vartheta_i) \circ g_i^{-1}$ acts on $\mathbf{S}^{n-1}$. We may assume without loss of generality that the the sequence of $\mathbf{d}$-isometries g_i converges to an isometry g of $\mathbf{S}^n$. Thus, $h(\vartheta_i) \circ g_i^{-1}(g_i(\mathbf{dev}(W(p_i))))$ converges to x', and $g_i \circ h(\vartheta_i)^{-1}(\mathbf{dev}(F_1))$ converges to $g(F_{1,\infty})$ by what we required in the beginning of the pull-back process. By Proposition B.1, we see that $g(F_{1,\infty})$ is an $(n-1)$-hemisphere, and we are done.

CHAPTER 5

THE TRANSVERSAL INTERSECTION OF n-CRESCENTS

From now on, we will assume that M is compact and with totally geodesic or empty boundary. We will discuss about the transversal intersection of n-crescents, generalizing that of crescents in two-dimensions [**10**].

First, we will show that if two hemispheric n-crescents overlap, then they are equal. For transversal intersection of two bihedral n-crescents, we will follow Section 2.6 of [**10**].

Our principal assumption is that M_h is not projectively diffeomorphic to an open n-hemisphere or n-bihedron, which will be sufficient for the results of this section to hold (see Remark 4.7). This is equivalent to requiring that $\tilde{M}$ is not projectively diffeomorphic to these. *This will be our assumption in Chapters 5 to 8.* In applying the results of these Chapters in Chapters 9 and 10 we need this assumption also.

For the following theorem, we may even relax this condition even further:

Theorem 5.1. — *Suppose that M_h is not projectively diffeomorphic to an open hemisphere. Suppose that R_1 and R_2 are two overlapping n-crescents that are hemispheres. Then $R_1 = R_2$, and hence $\nu_{R_1} = \nu_{R_2}$ and $\alpha_{R_1} = \alpha_{R_2}$.*

Proof. — We use Lemma 5.2 as in [**10**]: By Proposition 3.9, $\mathbf{dev}|R_1 \cup R_2$ is an imbedding onto the union of two n-hemispheres $\mathbf{dev}(R_1)$ and $\mathbf{dev}(R_2)$ in $\mathbf{S}^n$. If R_1 is not equal to R_2, then $\mathbf{dev}(R_1)$ differs from $\mathbf{dev}(R_2)$, $\mathbf{dev}(R_1)$ and $\mathbf{dev}(R_2)$ meet each other in a convex n-bihedron, $\mathbf{dev}(R_1)\cup\mathbf{dev}(R_2)$ is homeomorphic to an n-ball, and the boundary $\delta(\mathbf{dev}(R_1) \cup \mathbf{dev}(R_2))$ is the union of two $(n-1)$-hemispheres meeting each other in a great $(n-2)$-sphere $\mathbf{S}^{n-2}$.

Since α_{R_1} and α_{R_2} are disjoint from any of R_1^o and R_2^o respectively, the images of α_{R_1} and α_{R_2} do not intersect any of $\mathbf{dev}(R_1^o)$ and $\mathbf{dev}(R_2^o)$ respectively by Proposition 3.9. Therefore, $\mathbf{dev}(\alpha_{R_1})$ and $\mathbf{dev}(\alpha_{R_2})$ are subsets of $\delta(\mathbf{dev}(R_1) \cup \mathbf{dev}(R_2))$. Since they are open $(n-1)$-hemispheres, the complement of $\mathbf{dev}(\alpha_{R_1}) \cup \mathbf{dev}(\alpha_{R_2})$ in $\delta(\mathbf{dev}(R_1)\cup\mathbf{dev}(R_2))$ equals $\mathbf{S}^{n-2}$, and $\mathbf{dev}(\alpha_{R_1})\cup\mathbf{dev}(\alpha_{R_2})$ is dense in $\delta(\mathbf{dev}(R_1)\cup \mathbf{dev}(R_2))$. Since $\mathbf{dev}|R_1 \cup R_2$ is an imbedding, it follows that $R_1 \cup R_2$ is an n-ball, and the closure of $\alpha_{R_1} \cup \alpha_{R_2}$ equals $\delta(R_1 \cup R_2)$. Hence, $\delta(R_1 \cup R_2) \subset M_{h,\infty}$. By

Lemma 3.17 it follows that $M_h = R_1^o \cup R_2^o$, and M_h is boundaryless. By Lemma 5.2, this is a contradiction. Hence, $R_1 = R_2$. $\square$

Lemma 5.2. — *Let N be a closed real projective n-manifold. Suppose that* $\mathbf{dev}$: $\check{N}_h \to \mathbf{S}^n$ *is an imbedding onto the union of n-hemispheres H_1 and H_2 meeting each other in an n-bihedron or an n-hemisphere. Then $H_1 = H_2$, and N_h is projectively diffeomorphic to an open n-hemisphere.*

Proof. — Let $(\mathbf{dev}, h)$ denote the development pair of N, and Γ the deck transformation group. As $\mathbf{dev}|N_h$ is a diffeomorphism onto $H_1^o \cup H_2^o$, a simply connected set, we have $N_h = \tilde{N}$.

Suppose that $H_1 \neq H_2$. Then $H_1 \cup H_2$ is bounded by two $(n-1)$-hemispheres D_1 and D_2 meeting each other on a great sphere $\mathbf{S}^{n-2}$, their common boundary. Since the interior angle of intersection of D_1 and D_2 is greater than π, $\delta H_i - D_i$ is an open hemisphere included in $\mathbf{dev}(\tilde{N})$ for $i = 1, 2$. Defining $O_i = \delta H_i - D_i$ for $i = 1, 2$, we see that $O_1 \cup O_2$ is $h(\Gamma)$-invariant since $\delta(H_1 \cup H_2)$ is $h(\Gamma)$-invariant. This means that the inverse image $\mathbf{dev}^{-1}(O_1 \cup O_2)$ is Γ-invariant.

Let $O_i' = \mathbf{dev}^{-1}(O_i)$. Then elements of Γ either act on each of O_1' and O_2' or interchange them. Thus, Γ includes a subgroup Γ' of index one or two acting on each of O_1' and O_2'. Since N_h is a simply connected open ball, and so is O_1', it follows that the n-manifold $\tilde{N}/\Gamma'$ and an $(n-1)$-manifold O_1'/Γ' are homotopy equivalent. Since $\tilde{N}/\Gamma'$ is a finite cover of a closed manifold N, $\tilde{N}/\Gamma'$ is a closed manifold. Since the dimensions of $\tilde{N}/\Gamma'$ and O_1'/Γ are not the same, this is shown to be absurd by computing $\mathbf{Z}_2$-homologies. Hence we obtain that $H_1 = H_2$, and since $\mathbf{dev}(\tilde{N})$ equals the interior of H_1, $\tilde{N}$ is diffeomorphic to an open n-hemisphere. $\square$

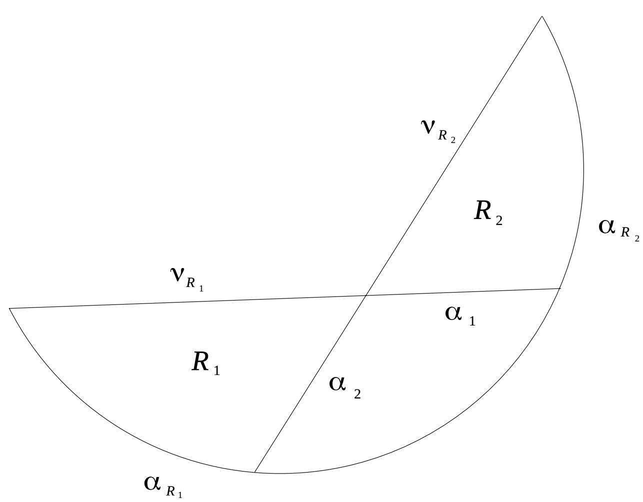

FIGURE 5.1. Transversal intersections in dimension two.

Suppose that R_1 is an n-crescent that is an n-bihedron. Let R_2 be another bihedral n-crescents with sets α_{R_2} and ν_{R_2}. We say that R_1 and R_2 intersect *transversally* if R_1 and R_2 overlap and the following conditions hold ($i = 1, j = 2$; or $i = 2, j = 1$):

1. $\nu_{R_1} \cap \nu_{R_2}$ is an $(n-2)$-dimensional hemisphere.
2. For the intersection $\nu_{R_1} \cap \nu_{R_2}$ denoted by H, H is an $(n-2)$-hemisphere, H^o is a subset of the interior $\nu_{R_i}^o$, and $\mathbf{dev}(\nu_{R_i})$ and $\mathbf{dev}(\nu_{R_j})$ intersect transversally at $\mathbf{dev}(H)$.
3. $\nu_{R_i} \cap R_j$ is a tame $(n-1)$-bihedron with boundary the union of H and an $(n-2)$-hemisphere H' in the closure of α_{R_j} with its interior H'^o in α_{R_j}.
4. $\nu_{R_i} \cap R_j$ is the closure of a component of $\nu_{R_i} - H$ in $\check{M}_h$.
5. $R_i \cap R_j$ is the closure of a component of $R_j - \nu_{R_i}$.
6. Both $\alpha_{R_i} \cap \alpha_{R_j}$ and $\alpha_{R_i} \cup \alpha_{R_j}$ are homeomorphic to open $(n-1)$-dimensional balls, which are locally totally geodesic under $\mathbf{dev}$.

Note that since α_{R_i} is tame, $\alpha_{R_i} \cap \alpha_{R_j}$ is tame. (See Figures 5.1 and 5.2.)

By Corollary 5.8, the above condition mirrors the property of intersection of $\mathbf{dev}(R_1)$ and $\mathbf{dev}(R_2)$ where $\mathbf{dev}(\alpha_{R_1})$ and $\mathbf{dev}(\alpha_{R_2})$ are included in a common great sphere $\mathbf{S}^{n-1}$ of dimension $(n-1)$, $\mathbf{dev}(R_1)$ and $\mathbf{dev}(R_2)$ included in a common n-hemisphere bounded by $\mathbf{S}^{n-1}$ and $\mathbf{dev}(\nu_{R_1})^o$ and $\mathbf{dev}(\nu_{R_2})^o$ meet transversally (see Proposition 3.9). Conversely, if the images of R_1 and R_2 satisfy these conditions, and R_1 and R_2 overlap, then R_1 and R_2 intersect transversally.

Example 5.3. — In the example 3.12, R is an n-crescent with the closure of the plane P given by the equation $x_1 + \cdots + x_n = 0$ equal to ν_R. α_R equals the interior of the intersection of R with δH. ν_S is the closure of the plane given by $x_1 = 0$ and α_S the interior of the intersection of S with δH. Clearly, R and S intersect transversally.

Using the reasoning similar to Section 2.6. of [**10**], we obtain:

Theorem 5.4. — *Suppose that R_1 and R_2 are overlapping. Then either R_1 and R_2 intersect transversally or $R_1 \subset R_2$ or $R_2 \subset R_1$.*

Remark 5.5. — In case R_1 is a proper subset of R_2, we see easily that $\alpha_{R_1} = \alpha_{R_2}$ since the sides of R_1 in $M_{h,\infty}$ must be in one of R_2. Hence, we also see that $\nu_{R_1}^o \subset R_2^o$ as the topological boundary of R_1 in R_2 must lie in ν_{R_1}.

The proof is entirely similar to that in [**10**]. A heuristic reason that the theorem holds is as follows (due to the referee): We show that $\mathbf{dev}|R_1 \cup R_2$ is an imbedding onto $\mathbf{dev}(R_1) \cup \mathbf{dev}(R_2)$ as R_1 and R_2 overlap. As α_{R_1} and α_{R_2} are subsets of $M_{h,\infty}$, their images $\mathbf{dev}(\alpha_{R_1})$ and $\mathbf{dev}(\alpha_{R_2})$ are disjoint from $\mathbf{dev}(R_1)^o$ and $\mathbf{dev}(R_2)^o$. Among all combinatorial types of configurations of bihedrons $\mathbf{dev}(R_1)$ and $\mathbf{dev}(R_2)$ in $\mathbf{S}^n$, the configuration satisfying this condition should be the one described above. For reasons of rigor, we present the following somewhat nonintuitive arguments; however, a parallel argument purely based on the images of R_1 and R_2 in $\mathbf{S}^n$ is also possible.

Assume that we have $i = 1$ and $j = 2$ or have $i = 2$ and $j = 1$, and $R_1 \not\subset R_2$ and $R_2 \not\subset R_1$. Since $\mathrm{Cl}(\alpha_{R_i}) \subset M_{h,\infty}$, Corollary 3.11 and Proposition 3.9 imply that R_j

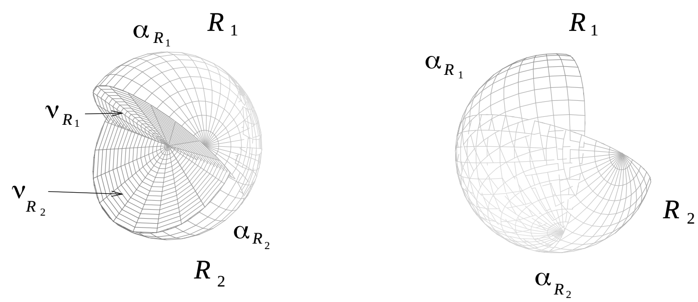

FIGURE 5.2. A three-dimensional transversal intersection seen in two view points

dips into (R_i, ν_{R_i}), *a point of view we shall hold for a while.* Hence, the following statements hold:

– $\nu_{R_i} \cap R_j$ is a convex $(n-1)$-ball α_i such that

$$\delta\alpha_i \subset \delta R_j, \alpha_i^o \subset R_j^o. \tag{5.1}$$

– $R_i \cap R_j$ is the convex n-ball that is the closure of a component of $R_j - \alpha_i$.

Since α_i^o is disjoint from ν_{R_j}, α_i^o is a subset of a component C of $\nu_{R_i} - \nu_{R_j}$.

Lemma 5.6. — *If ν_{R_i} and ν_{R_j} meet, then they do so transversally; i.e, their images under* **dev** *meet transversally. If ν_{R_i} and α_{R_j} meet, then they do so transversally.*

Proof. — Suppose that ν_{R_i} and ν_{R_j} meet and they are tangential. Then $\mathbf{dev}(\nu_{R_i})$ and $\mathbf{dev}(\nu_{R_j})$ both lie on a common great $(n-1)$-sphere in $\mathbf{S}^n$. Since $\mathbf{dev}(R_j)$ lies in an n-hemisphere bounded by this sphere, $\nu_{R_i} \cap \nu_{R_j} = \nu_{R_i} \cap R_j$ by Proposition 3.9. Since $\nu_{R_i} \cap R_j$ includes an open $(n-1)$-ball α_i^o, this contradicts $\alpha_i^o \subset R_j^o$.

Suppose that ν_{R_i} and α_{R_j} meet and they are tangential. Then $\nu_{R_i} \cap \mathrm{Cl}(\alpha_{R_j}) = \nu_{R_i} \cap R_j$ as before, which leads to contradiction similarly. □

We now determine a preliminary property of α_i. Since α_i is a convex $(n-1)$-ball in ν_{R_i} with topological boundary in $\delta R_i \cup \delta R_j$, we obtain

$$\begin{aligned} \delta\alpha_i &\subset \delta\nu_{R_i} \cup (\delta R_j \cap \nu_{R_i}^o) \\ &\subset \delta\nu_{R_i} \cup (\nu_{R_j} \cap \nu_{R_i}^o) \cup (\alpha_{R_j} \cap \nu_{R_i}^o). \end{aligned}$$

Hence, we have

$$\delta\alpha_i = (\delta\alpha_i \cap \delta\nu_{R_i}) \cup (\delta\alpha_i \cap \nu_{R_j} \cap \nu^o_{R_i}) \cup (\delta\alpha_i \cap \alpha_{R_j} \cap \nu^o_{R_i}).$$

If α_{R_j} meets $\nu^o_{R_i}$, then since α_{R_j} is transversal to $\nu^o_{R_i}$ by Lemma 5.6, α_{R_j} must intersect R^o_i by Proposition 3.9. Since $\alpha_{R_j} \subset M_{h,\infty}$, this is a contradiction. Thus, $\alpha_{R_j} \cap \nu^o_{R_i} = \emptyset$. We conclude

$$\delta\alpha_i = (\delta\alpha_i \cap \delta\nu_{R_i}) \cup (\delta\alpha_i \cap \nu_{R_j} \cap \nu^o_{R_i}). \tag{5.2}$$

Before continuing the proof, let us state a few easy spherical geometry facts: Given an i-hemisphere J in $\mathbf{S}^n$, if K is a great sphere in J, then K is included in δJ. Thus if K is a compact convex set containing a point of J^o, then K must be an j-ball for some j, $0 \leq j \leq i$, by Proposition 2.6.

An $(i-1)$-hemisphere K, $i \geq 1$, in J has its boundary in δJ. If K contains a point of J^o, then K equals $J \cap L$ for a great $(i-1)$-sphere L, and $J^o - K$ has exactly two components which are convex. Since the closures of the components includes K, Lemma 2.5 shows that the closures of two components are i-bihedrons.

Suppose that K is a convex $(i-1)$-ball in J meeting J^o. Then K must be a convex subset of $J \cap L$ for a great $(i-1)$-sphere L. If δK meets J^o, then $J \cap L$ is an $(i-1)$-hemisphere which includes K as a proper convex subset. Thus $J^o - K$ can have only one component. Hence, if K meets J^o, and $J^o - K$ has at least two components, then K must be an $(i-1)$-hemisphere. In this case, it is obvious that $K^o \subset J^o$.

Let us denote by H the set $\nu_{R_i} \cap \nu_{R_j}$. Consider for the moment the case where $\nu_{R_j} \cap \nu^o_{R_i} \neq \emptyset$. Since $\mathbf{dev}(H)$ is a compact convex set and is included in an $(n-1)$-hemisphere $\mathbf{dev}(\nu_{R_i})$, Lemma 5.6 and above paragraphs show that $\mathbf{dev}(H)$ is a compact convex $(n-2)$-ball. Thus, H is a tame topological $(n-2)$-ball by Proposition 3.9.

If H has boundary points, i.e, points of δH, in $\nu^o_{R_i}$, then $\nu^o_{R_i} - H$ would have only one component. Since the boundary of α_i in $\nu^o_{R_i}$ is included in H by equation 5.2, α^o_i is dense in ν_{R_i}, implying $\alpha_i = \nu_{R_i}$. Since α^o_i is a subset of R^o_j by equation 5.1, $\nu^o_{R_i}$ is a subset of R^o_j, which contradicts our momentary assumption. It follows that H is an $(n-2)$-hemisphere with boundary in $\delta\nu_{R_i}$ and the interior H^o in $\nu^o_{R_i}$, and H separates ν_{R_i} into two convex components, and the closures of each of them are $(n-1)$-bihedrons. Obviously, α_i is the closure of one of the components.

We need to consider only the following two cases by interchanging i and j if necessary:

(i) $\nu^o_{R_j} \cap \nu^o_{R_i} \neq \emptyset$.
(ii) $\nu_{R_j} \cap \nu^o_{R_i} = \emptyset$ or $\nu_{R_i} \cap \nu^o_{R_j} = \emptyset$.

(i) Since α_i is the closure of a component of $\nu_{R_i} - H$, α_i is an $(n-1)$-bihedron bounded by an $(n-2)$-hemisphere H and another $(n-2)$-hemisphere H' in $\delta\nu_{R_i}$. Since H' is a subset of the closure of α_{R_i}, H' is a subset of $M_{h,\infty}$ and hence disjoint from R^o_j while $R^o_j \subset M_h$.

Since H' is a subset of R_j, we have $H' \subset \delta R_j$. Since ν_{R_i} is transversal to ν_{R_j}, H' is not a subset of ν_{R_j}; thus, H'^o is a subset of α_{R_j}, and H' that of $\mathrm{Cl}(\alpha_{R_j})$ by Proposition 3.9. This completes the proof of the transversality properties (1)–(4) in case (i).

(5) By dipping intersection properties, $R_i \cap R_j$ is the closure of a component of $R_j - \alpha_i$ and hence that of $R_j - \nu_{R_i}$.

(6) Since H'^o is a subset of α_{R_j}, $\alpha_{R_j} - H'$ has two components β_1 and β_2, homeomorphic to open $(n-1)$-balls. By (5), we may assume without loss of generality that β_1 is a subset of R_i, and β_2 is disjoint from R_i. Since $\beta_1 \subset M_{h,\infty}$, we have $\beta_1 \subset \delta R_i$. As β_1 is a component of α_{R_j} removed with H', we see that β_1 is an open $(n-1)$-bihedron bounded by H' in $\delta\nu_{R_i}$ and an $(n-2)$-hemisphere H'' in $\delta\nu_{R_j}$ (see Figure 5.3).

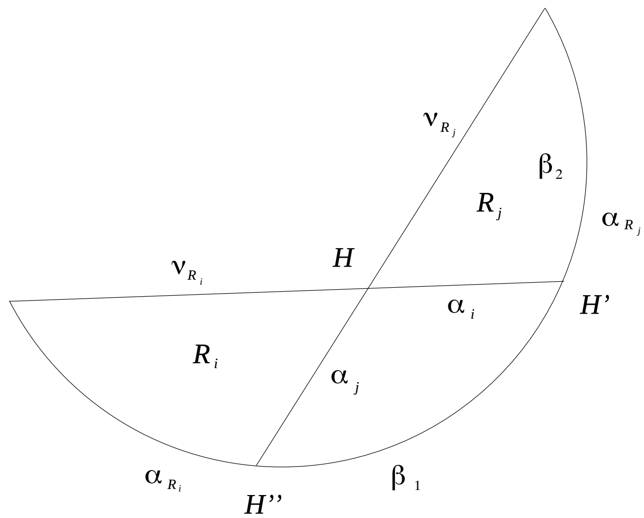

FIGURE 5.3. A figure to explain (i)(6).

Since (i) holds for i and j exchanged, we obtain, by a paragraph above the condition (i), H^o belongs to $\nu^o_{R_i} \cap \nu^o_{R_j}$.

Since the closure of β_1 belongs to R_i, we obtain that $H'' \subset R_i$ and H'' is a subset of α_j, where $\alpha_j = \nu_{R_j} \cap R_i$. As H'' is a subset of $\delta\nu_{R_j}$, and α_j is the closure of a component of ν_{R_j} removed with H, we obtain $H'' \subset \delta\alpha_j$.

By (1)–(4) with values of i and j exchanged, α_j is an $(n-1)$-bihedron bounded by H and an $(n-2)$-hemisphere H''' with interior in α_{R_i} and is the closure of a component of $\nu_{R_j} - H$. Since H'' is an $(n-2)$-hemisphere in $\delta\alpha_j$, and so is H''', it follows that $H'' = H'''$.

Since β_1 has the boundary the union of H' in $\delta\nu_{R_i}$ and H'', $H'' = H'''$, with interior in α_{R_i}, and β_1 is a convex subset of R_i, looking at the bihedron $\mathbf{dev}(R_i)$ and the geometry of $\mathbf{S}^n$ show that $\beta_1 \subset \alpha_{R_i}$. Thus, we obtain $\beta_1 \subset \alpha_{R_i} \cap \alpha_{R_j}$. We see that $\mathbf{dev}(\alpha_{R_i})$ and $\mathbf{dev}(\alpha_{R_j})$ are subsets of a common great $(n-1)$-sphere; it follows easily by Proposition 3.9 that $\beta_1 = \alpha_{R_i} \cap \alpha_{R_j}$. Hence, $\alpha_{R_i} \cap \alpha_{R_j}$ and $\alpha_{R_i} \cup \alpha_{R_j}$ are homeomorphic to open $(n-1)$-balls, and under $\mathbf{dev}$ they map to totally geodesic $(n-1)$-balls in $\mathbf{S}^n$.

(ii) Assume $\nu^o_{R_j} \cap \nu_{R_i} = \emptyset$ without loss of generality. Then H is a subset of $\delta\nu_{R_j}$. Since R_i dips into (R_j, ν_{R_j}), we have that $\alpha_j \neq \emptyset$. Since the boundary of α_j in $\nu^o_{R_j}$

is included in H (see equation 5.2), we have $\alpha_j = \nu_{R_j}$ and $\nu_{R_j} \subset R_i$. Since ν_{R_j} is an $(n-1)$-hemisphere, and R_i is an n-bihedron, the uniqueness of $(n-2)$-spheres in an n-bihedron (Proposition 2.4) shows that $\delta\nu_{R_i} = \delta\nu_{R_j}$. Thus, the closures of components of $R_i - \nu_{R_j}$ are n-bihedrons with respective boundaries $\alpha_{R_i} \cup \nu_{R_j}$ and $\nu_{R_i} \cup \nu_{R_j}$. By Corollary 3.11, $R_i \cap R_j$ is the closure of either the first component or the second one.

In the first case, $R_i^o \cap R_j^o$ is an open subset of R_j^o since R_i^o is open in M_h^o. The closure of R_i^o in M_h equals $R_i^o \cup (\nu_{R_i} \cap M_h) = R_i \cap M_h$. Since $\nu_{R_i}^o$, which includes $\nu_{R_i} \cap M_h$, does not meet R_j in the first case being in the other component of $R_i - \nu_{R_j}$, we see that the intersection of the closure of R_i^o in M_h with R_j^o is same as $R_i^o \cap R_j^o$. Thus, $R_i^o \cap R_j^o$ is open and closed subset of R_j^o. Hence $R_j^o \subset R_i^o$ and $R_i \subset R_j$. This contradicts our hypothesis.

In the second case, $\mathbf{dev}|R_i \cup R_j$ is a homeomorphism to $\mathbf{dev}(R_i) \cup \mathbf{dev}(R_j)$. As α_{R_i} and α_{R_j} are subsets of $M_{h,\infty}$, their images under $\mathbf{dev}$ may not meet that of $R_i^o \cup R_j^o$. Hence, $\mathbf{dev}(R_i) \cup \mathbf{dev}(R_j)$ is an n-ball bounded by two $(n-1)$-hemispheres $\mathbf{dev}(\mathrm{Cl}(\alpha_{R_i}))$ and $\mathbf{dev}(\mathrm{Cl}(\alpha_{R_j}))$. We obtain that $R_i \cup R_j$ is the n-ball bounded by two $(n-1)$-dimensional hemispheres $\mathrm{Cl}(\alpha_{R_i})$ and $\mathrm{Cl}(\alpha_{R_j})$.

Since $\mathrm{Cl}(\alpha_{R_i})$ and $\mathrm{Cl}(\alpha_{R_j})$ are subsets of $M_{h,\infty}$, Lemma 3.17 shows that $\check{M}_h = R_i \cup R_j$ and $M_h = R_i^o \cup R_j^o$; thus, $M_h = \tilde{M}$ and M is a closed manifold. The image $\mathbf{dev}(R_1) \cup \mathbf{dev}(R_2)$ is bounded by two $(n-1)$-hemispheres meeting each other on a great sphere $\mathbf{S}^{n-2}$, their common boundary. Since M_h is not projectively diffeomorphic to an open n-hemisphere or an open n-bihedron, the interior angle of intersection of the two boundary $(n-1)$-hemisphere should be greater than π. However, Lemma 5.2 contradicts this. ☐

Remark 5.7. — Using the same proof as above, we may drop the condition on the Euler characteristic from Theorem 2.6 of [**10**] if we assume that $\tilde{M}$ is not projectively diffeomorphic to an open 2-hemisphere or an open lune. This is weaker than requiring that the Euler characteristic of M is less than zero. So, our theorem is an improved version of Theorem 2.6 of [**10**].

Corollary 5.8. — *Let R_1 and R_2 be bihedral n-crescents and they overlap. Then the following statements hold*:

- $\mathbf{dev}(\alpha_{R_1})$ *and* $\mathbf{dev}(\alpha_{R_2})$ *are included in a common great $(n-1)$-sphere $\mathbf{S}^{n-1}$,*
- $\mathbf{dev}(\nu_{R_1})$ *and* $\mathbf{dev}(\nu_{R_2})$ *meet in an $(n-2)$-hemisphere transversally,*
- $\mathbf{dev}(R_1)$ *and* $\mathbf{dev}(R_2)$ *are subsets of a common great n-hemisphere bounded by $\mathbf{S}^{n-1}$.*
- $\mathbf{dev}(R_i - \mathrm{Cl}(\alpha_{R_i}))$ *is a subset of the interior of this n-hemisphere for $i = 1, 2$.*

☐

CHAPTER 6

HEMISPHERIC n-CRESCENTS AND TWO-FACED SUBMANIFOLDS

In this chapter, we introduce the two-faced submanifolds arising from hemispheric n-crescents. We showed above that if two hemispheric n-crescents overlap, then they are equal. We show that if two hemispheric n-crescents meet but do not overlap, then they meet at the union of common components of their ν-boundaries, which we call copied components. The union of all copied components becomes a properly imbedded submanifold in M_h and covers a closed submanifold in M, which is said to be the two-faced submanifold.

Lemma 6.1. — *Let R be an n-crescent. A component of δM_h is either disjoint from R or is a component of $\nu_R \cap M_h$. Moreover, a tiny ball $B(x)$ of a point x of δM_h is a subset of R if x belongs to $\nu_R \cap M_h$, and, consequently, x belongs to the topological interior* $\mathrm{int} R$.

Proof. — If $x \in \delta M_h$, then a component F of the open $(n-1)$-manifold $\nu_R \cap M_h$ intersects δM_h tangentially, and by Lemma 3.1, it follows that F is a subset of δM_h. Since F is a closed subset of $\nu_R \cap M_h$, F is a closed subset of δM_h. Since F is an open manifold, F is open in δM_h. Thus, F is a component of δM_h.

Since $x \in \mathrm{int} B(x)$, $B(x)$ and R overlap. As $\mathrm{Cl}(\alpha_R)$ is a subset of $M_{h,\infty}$, we have $\mathrm{bd} R \cap B(x) \subset \nu_R$ and $\nu_R \cap B(x) = F \cap B(x)$ for a component F of $\nu_R \cap M_h$ containing x. Since F is a component of δM_h, we obtain $F \cap B(x) \subset \delta B(x)$; since we have $\mathrm{bd} R \cap B(x) \subset \delta B(x)$, it follows that $B(x)$ is a subset of R. $\square$

Suppose that $\check{M}_h$ includes an n-crescent R that is an n-hemisphere. Then $M_h \cap R$ is a submanifold of M_h with boundary $\delta R \cap M_h$. Since R is an n-crescent, $\delta R \cap M_h$ equals $\nu_R \cap M_h$. Let B_R denote $\nu_R \cap M_h$.

Let S be another hemispheric n-crescent, and B_S the set $\nu_S \cap M_h$. By Theorem 5.1, we see that either $S \cap R^o = \emptyset$ or $S = R$. Suppose that $S \cap R \neq \emptyset$ and S does not equal R. Then $B_S \cap B_R \neq \emptyset$. Let x be a point of $B_S \cap B_R$ and $B(x)$ the tiny ball of x. Since $\mathrm{int} B(x) \cap R \neq \emptyset$, it follows that $B(x)$ dips into (R, ν_R) or $B(x)$ is a subset of R by Lemma 3.13. Similarly, $B(x)$ dips into (S, ν_S) or $B(x)$ is a subset of S. If $B(x)$ is a subset of R, then S intersects the interior of R. Theorem 5.1 shows $S = R$, a contradiction. Therefore, $B(x)$ dips into (R, ν_R) and similarly into (S, ν_S). If ν_R

and ν_S intersect transversally, then R and S overlap, implying a contradiction $S = R$. Therefore, B_S and B_R intersect tangentially at x.

If $x \in \delta M_h$, Lemma 6.1 shows that $B(x)$ is a subset of R. This contradicts a result of the above paragraph. Thus, $x \in M_h^o$. Hence, we conclude that $B_R \cap B_S \subset M_h^o$.

Since B_R and B_S are closed subsets of M_h, and B_R and B_S are totally geodesic and intersect tangentially at x, it follows that $B_R \cap B_S$ is an open and closed subset of B_R and B_S respectively. Thus, for components A of B_R and B of B_S, either we have $A = B$ or A and B are disjoint. Therefore, we have proved:

Proposition 6.2. — *Given two hemispheric n-crescents R and S, we have either R and S disjoint, or R equals S, or $R \cap S$ equals the union of common components of $\nu_R \cap M_h$ and $\nu_S \cap M_h$ in M_h^o.*

Readers may easily find examples where $\nu_R \cap M_h$ and $\nu_S \cap M_h$ are not equal in the above situations.

Definition 6.3. — Given a hemispheric n-crescent T, we say that a component of $\nu_T \cap M_h$ is *copied* if it equals a component of $\nu_U \cap M_h$ for some hemispheric n-crescent U not equal to T.

Let c_R be the union of all copied components of $\nu_R \cap M_h$ for a hemispheric n-crescent R. Let A denote $\bigcup_{R \in \mathcal{H}} c_R$ where $\mathcal{H}$ is the set of all hemispheric n-crescents in M_h. A is said to be the *pre-two-faced submanifold arising from hemispheric n-crescents.*

Proposition 6.4. — *Suppose that A is not empty. Then A is a properly imbedded totally geodesic $(n-1)$-submanifold of M_h^o and $p|A$ is a covering map onto a closed totally geodesic imbedded $(n-1)$-manifold in M^o.*

First, given two n-crescents R and S, c_R and c_S meet either in the union of common components or in an empty set: Let a and b be respective components of c_R and c_S meeting each other. Then a is a component of $\nu_R \cap M_h$ and b that of $\nu_S \cap M_h$. Since $R \cap S \neq \emptyset$, either R and S overlap or $a = b$ by the above argument. If R and S overlap, $R = S$ and hence a and b must be the identical component of $\nu_R \cap M_h$ and hence $a = b$. Therefore, A is a union of mutually disjoint closed path-components that are components of c_R for some n-crescent R.

Second, given a tiny ball $B(x)$ of a point x of M_h, we claim that no more than one path-component of A may intersect $\mathrm{int} B(x)$: Let a be a component of c_R intersecting $\mathrm{int} B(x)$. Since copied components are subsets of M_h^o, a intersects $B(x)^o$ and hence $B(x)$ is not a subset of R. By Lemma 3.13, $\nu_R \cap B(x)$ is a compact convex $(n-1)$-ball with boundary in $\mathrm{bd} B(x)$. Since it is connected, $a \cap B(x) = \nu_R \cap B(x)$, and $B(x) \cap R$ is the closure of a component C_1 of $B(x) - (a \cap B(x))$ by Corollary 3.11. Since a is copied, a is a component of $\nu_S \cap M_h$ for an n-crescent S not equal to R, and $B(x) \cap S$ is the closure of a component C_2 of $B(x) - (a \cap B(x))$. Since R and S do not overlap, it follows that C_1 and C_2 are the two disjoint components of $B(x) - (a \cap B(x))$.

Suppose that b is a component of c_T for an n-crescent T and b intersects $\mathrm{int} B(x)$ also. If the $(n-1)$-ball $b \cap B(x)$ intersects C_1 or C_2, then T overlaps R or S respectively and hence $T = R$ or $T = S$ respectively by Theorem 5.1; therefore, we have $a = b$. This is absurd. Hence $b \cap B(x) \subset a \cap B(x)$ and T overlaps with either R or S.

Since these are hemispheric crescents, we have either $T = R$ or $T = S$ respectively; therefore, $a = b$. We conclude that if $\mathrm{int} B(x) \cap A$ is not empty, then $B(x) \cap A$ equals a compact $(n-1)$-ball with boundary in $\mathrm{bd} B(x)$.

Since each path-component of A is an open subset of A, the above shows that A is a totally geodesic $(n-1)$-submanifold of $\check{M}_h^o$, closed and properly imbedded in $\check{M}_h^o$.

Let $p : \check{M}_h \to M$ be the covering map. Since A is the deck transformation group invariant, we have $A = p^{-1}(p(A))$ and $p|A$ covers $p(A)$. The above results show that $p(A)$ is a closed totally geodesic manifold in M^o.

Definition 6.5. — The image $p(A)$ for the union A of all copied components of hemispheric n-crescents in $\check{M}_h$ is said to be the *two-faced* $(n-1)$*-manifold* of M *arising from hemispheric* n*-crescents* (or *type* I).

Each component of $p(A)$ is covered by a component of A, i.e., a copied component of $\nu_R \cap \check{M}_h$ for some crescent R. Since α_R is the union of the open $(n-1)$-hemispheres in δR, $\nu_R \cap \check{M}_h$ lies in an open $(n-1)$-hemisphere, i.e., an affine patch in the great $(n-1)$-sphere δR. Hence, each component of $p(A)$ is covered by an open domain in $\mathbf{R}^n$, as we mentioned in the introduction.

We end with the following observation:

Proposition 6.6. — *Suppose that* $A = \bigcup_{R \in \mathcal{H}} c_R$. *Then* A *is disjoint from* S^o *for each hemispheric* n*-crescent* S *in* $\check{M}_h$.

Proof. — If not, then a point x of c_R meets S^o for some hemispheric n-crescent S. But if so, then R and S overlap, and $R = S$, a contradiction. $\square$

CHAPTER 7

BIHEDRAL n-CRESCENTS AND TWO-FACED SUBMANIFOLDS

In this chapter, we will define an equivariant set $\Lambda(R)$ for a bihedral n-crescent R, which will play the role of hemispheric n-crescents in the previous chapter. We discuss its properties which are exactly same as those of its two-dimensional version in [**10**]. Then we discuss the two-faced submanifold that arises from $\Lambda(R)$'s: We show that $\Lambda(R)$ and $\Lambda(S)$ for two n-crescents are either equal or disjoint or meet at their common boundary components in M_h. The union of all such boundary components for $\Lambda(R)$ for every bihedral n-crescent R is shown to be a totally geodesic properly imbedded submanifold in M_h and cover a closed totally geodesic submanifold of M.

We will suppose in this chapter that $\check{M}_h$ includes no hemispheric crescent; i.e., we assume that all n-crescents in $\check{M}_h$ are bihedrons. Two bihedral n-crescents in $\check{M}_h$ are equivalent if they overlap. This generates an equivalence relation on the collection of all bihedral n-crescents in $\check{M}_h$; that is, $R \sim S$ if and only if there exists a sequence of bihedral n-crescents R_i, $i = 1, \ldots, n$, such that $R_1 = R, R_n = S$ and $R_{i-1} \cap R_i^o \neq \emptyset$ for $i = 2, \ldots, n$.

We define

$$\Lambda(R) := \bigcup_{S \sim R} S, \qquad \delta_\infty \Lambda(R) := \bigcup_{S \sim R} \alpha_S, \qquad \Lambda_1(R) := \bigcup_{S \sim R} (S - \nu_R).$$

Example 7.1. — Consider the universal cover L of $H^o - \{O\}$ where H is a 2-hemisphere in $\mathbf{S}^2$. Then it has an induced real projective structure with developing map equal to the covering map c. There is a nice parameterization (r, θ) of L where r denotes the $\mathbf{d}$-distance of $c(x)$ from O and $\theta(x)$ the oriented total angle from the lift of the positive x-axis for $x \in L$, i.e., one obtained by integrating the 1-form lifted from the standard angular form $d\theta$ on the affine space H^o. Here r belongs to $(0, \pi/2)$ and θ to $(-\infty, \infty)$. L is hence a holonomy cover of itself as it is simply connected. $\check{L}$ may be identified with the universal cover of $H - \{O\}$ with a point O' added to make it a complete space where O' maps to O under the extended developing map c. (We use the universal covering space since the holonomy cover gives us uninteresting examples.)

We can determine that $\tilde{L}_\infty$ equals the union of $\{O'\}$ and a $\mathbf{d}$-infinite geodesic given by equation $\mathbf{d}(O', x) = \pi/2$, i.e., the inverse image of δH. A crescent in $\check{L}$ is the closure of a lift of an affine half space in $H^o - \{O\}$. A special type of a crescent is the closure of the set given by $\theta_0 \leq \theta \leq \theta_0 + \pi$. Given a crescent R in $\check{L}$, we see that $\Lambda(R)$ equals $\check{L}$.

We may also define another real projective manifold N by an equation $f(\theta) < r < \pi/2$ for a function f with values in $(0, \pi/2)$. Then $\check{N}$ equals the closure of N in $\check{L}$. Given a crescent R in $\check{N}$, we see that $\Lambda(R)$ may not equal to $\check{N}$ especially in case f is not a convex function (as seen in polar coordinates). (See Figure 7.1.)

For a higher dimensional example, let H be a 3-hemisphere in $\mathbf{S}^3$, and l a segment of $\mathbf{d}$-length π passing through the origin. Let L be the universal cover of $H^o - l$. Then L becomes a real projective manifold with developing map the covering map $c : L \to H^o - l$. The holonomy cover of L is L itself. The completion $\check{L}$ of L equals the completion of the universal cover of $H - l$ with l attached to make it a complete space. A 3-crescent is the closure of a lift of an open half space in $H - l$. Given a 3-crescent R, $\Lambda(R)$ equals $\check{L}$.

We introduce coordinates on H^o so that l^o is now the z-axis. Note that L is parameterized by (r, θ, ϕ) where $r(x)$ equals the $\mathbf{d}$-distance from O to $c(x)$, ϕ the angle that $\overline{Oc(x)}$ makes with the positive z-axis, and $\theta(x)$ again the integral of the obvious 1-form lifted from the standard angular form $d\theta$ in $\mathbf{R}^3$. We may also define other real projective manifolds by equation $f(\theta, \phi) < r < \pi/2$ for $f : \mathbf{R} \times (0, \pi) \to (0, \pi/2)$. The readers may work out how the completions might look and what $\Lambda(R)$ be when R is a 3-crescent. We remark that for certain f which converges to $\pi/2$ as $\phi \to 0$ or π, we may have no 3-crescents in the completion of the real projective manifold given by f.

Even higher-dimensional examples are given in a similar spirit by removing sets from such covers. After reading this section, the reader can easily see that these are really typical examples of $\Lambda(R)$.

Let us state the properties that hold for these sets: The proofs are straightforward and exactly as in [**10**].

$$\begin{aligned}
\mathrm{int}\Lambda(R) \cap M_h &= \mathrm{int}(\Lambda(R) \cap M_h) \\
\mathrm{bd}\Lambda(R) \cap M_h &= \mathrm{bd}(\Lambda(R) \cap M_h) \cap M_h
\end{aligned}$$

(see Lemma 6.4 [**10**]). For a deck transformation ϑ, from definitions we easily obtain

$$\begin{aligned}
\vartheta(\Lambda(R)) &= \Lambda(\vartheta(R)) \\
\vartheta(\delta_\infty\Lambda(R)) &= \delta_\infty\Lambda(\vartheta(R)) \\
\vartheta(\Lambda_1(R)) &= \Lambda_1(\vartheta(R)) \\
\mathrm{int}\vartheta(\Lambda(R)) \cap M_h &= \vartheta(\mathrm{int}\Lambda(R)) \cap M_h = \vartheta(\mathrm{int}\Lambda(R) \cap M_h) \\
\mathrm{bd}\vartheta(\Lambda(R)) \cap M_h &= \vartheta(\mathrm{bd}\Lambda(R)) \cap M_h = \vartheta(\mathrm{bd}\Lambda(R) \cap M_h).
\end{aligned} \tag{7.1}$$

The sets $\Lambda(R)$ and $\Lambda_1(R)$ are path-connected. $\delta_\infty\Lambda(R)$ is an open $(n-1)$-manifold. Since Theorem 5.4 shows that for two overlapping n-crescents R_1 and R_2, α_{R_1} and α_{R_2} extend each other into a larger $(n-1)$-manifold, there exists a unique great sphere $\mathbf{S}^{n-1}$ including $\mathbf{dev}(\delta_\infty\Lambda(R))$ and by Corollary 5.8, a unique component A_R

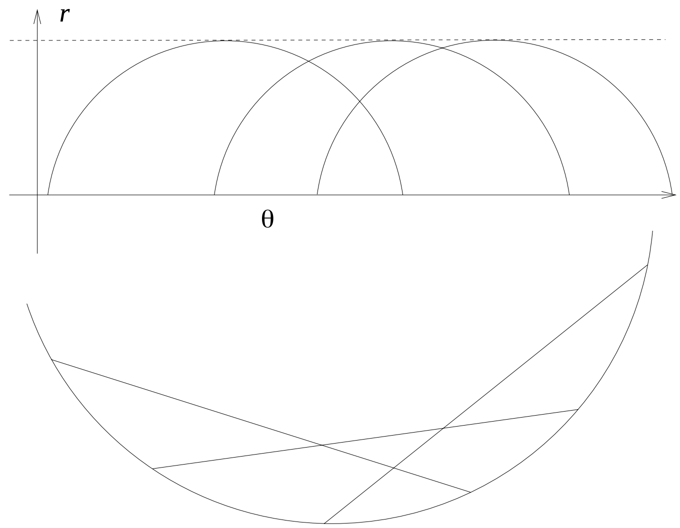

FIGURE 7.1. Figures of $\Lambda(R)$.

of $\mathbf{S}^n - \mathbf{S}^{n-1}$ such that $\mathbf{dev}(\Lambda(R)) \subset \mathrm{Cl}(A_R)$ and $\mathbf{dev}(\Lambda(R) - \mathrm{Cl}(\delta_\infty \Lambda(R))) \subset A_R$. For a deck transformation ϑ acting on $\Lambda(R)$, A_R is $h(\vartheta)$-invariant. $\Lambda_1(R)$ admits a real projective structure as a manifold with totally geodesic boundary $\delta_\infty \Lambda(R)$.

Proposition 7.2. — *$\Lambda(R) \cap M_h$ is a closed subset of M_h.*

Proof. — Lemma 9.2 implies this proposition. □

Lemma 7.3. — *$\mathrm{bd}\Lambda(R) \cap M_h$ is a properly imbedded topological submanifold of M_h^o, and $\Lambda(R) \cap M_h$ is a real projective submanifold of M_h with concave boundary $\mathrm{bd}\Lambda(R) \cap M_h$.*

Proof. — Let r be a point of $\mathrm{bd}\Lambda(R) \cap M_h$. Since $\Lambda(R)$ is closed, r is a point of a crescent R' equivalent to R. If r is a point of δM_h, then Lemma 6.1 implies that $r \in \mathrm{int}R'$ and $r \in \mathrm{int}\Lambda(R)$, a contradiction. Thus, $\mathrm{bd}\Lambda(R) \cap M_h \subset M_h^o$.

Let $B(r)$ be an open tiny ball of r. Since by Lemma 6.1, $\mathrm{bd}\Lambda(R) \cap M_h$ is a subset of M_h^o, $B(r)^o$ is an open neighborhood of r. Since $B(r)^o \cap \Lambda(R)$ is a closed subset of $B(r)^o$, $O = B(r)^o - \Lambda(R)$ is an open subset.

We claim that O is a convex subset of $B(r)^o$. Let $x, y \in O$. Then let s be the segment in $B(r)$ of $\mathbf{d}$-length $\leq \pi$ connecting x and y. If $s^o \cap \Lambda(R) \neq \emptyset$, then a point z of s^o belongs to an n-crescent S, $S \sim R$. If z belongs to S^o, since s must leave S, s meets ν_S and is transversal to ν_S at the intersection point. Since a maximal line in the bihedron S transversal to ν_S have an endpoint in α_S, at least one endpoint of s belongs to S^o, which is a contradiction. If z belongs to ν_S and s is transversal to ν_S at z, the same argument gives us a contradiction. If z belongs to ν_S and s is tangential to ν_S at z, then s is included in the component of $\nu_S \cap M_h$ containing z since $s \subset M_h$ is connected. Since x and y belong to O, this is a contradiction. Hence $s \subset O$, and O is convex.

Since O is convex and open, $\mathrm{bd}O$ in M_h is homeomorphic to an $(n-1)$-sphere by Proposition 2.6. As the boundary $\mathrm{bd}_{B(r)^o}O$ of O relative to $B(r)^o$ equals $\mathrm{bd}O \cap B(r)^o$, $\mathrm{bd}_{B(r)^o}O$ is an imbedded open $(n-1)$-submanifold of $B(r)^o$. While we have $\mathrm{bd}\Lambda(R) \cap B(r)^o = \mathrm{bd}_{B(r)^o}O$, $\mathrm{bd}\Lambda(R) \cap M_h$ is an imbedded $(n-1)$-submanifold. □

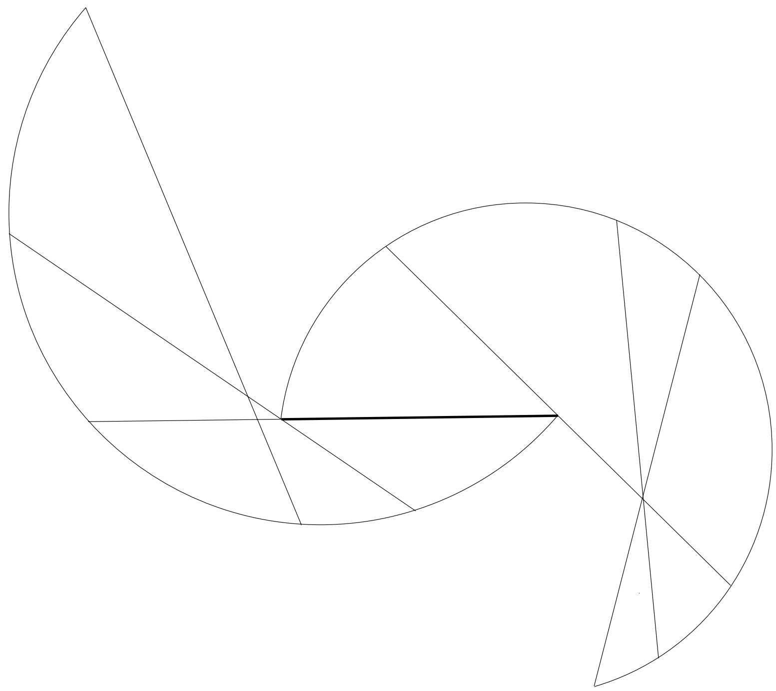

FIGURE 7.2. A pre-two-faced submanifold.

Using the same argument as in Section 6.2 of [**10**] (see Lemma 6.4 of [**10**]), we obtain the following lemma:

Lemma 7.4. — *If* $\mathrm{int}\Lambda(R) \cap M_h \cap \Lambda(S) \neq \emptyset$ *for an n-crescent S, then we have* $\Lambda(R) = \Lambda(S)$. *Moreover, if for a crescent S,* $\Lambda(R) \cap M_h$ *and* $\Lambda(S) \cap M_h$ *meet and they are distinct, then* $\Lambda(R)\cap\Lambda(S)\cap M_h$ *is a subset of* $\mathrm{bd}\Lambda(R)\cap M_h$ *and* $\mathrm{bd}\Lambda(S)\cap M_h$.

Proof. — If $\mathrm{int}(\Lambda(R) \cap M_h) \cap \Lambda(S) \neq \emptyset$ for an n-crescent S, then an n-crescent T, $T \sim S$, intersects $\mathrm{int}B(x)$ for $B(x) \subset \mathrm{int}\Lambda(R) \cap M_h$ where $B(x)$ is a tiny ball of a point x of $\mathrm{int}\Lambda(R) \cap M_h$.

If $B(x)$ is not a subset of T, then a component of $B(x) - a$ for an $(n-1)$-ball a, $a = \nu_T \cap B(x)$, with $\mathrm{bd}a \subset \mathrm{bd}B(x)$ is a subset of T^o by Lemma 3.13. Thus, whether $B(x)$ is a subset of T or not, a point y of $\mathrm{int}B(x)$ lies in T^o. Since y belongs to T' for some $T' \sim R$, it follows that T' and T overlap and hence $R \sim S$; therefore, $\Lambda(R) = \Lambda(S)$.

The second part follows easily from the first part. □

Assume now that $\Lambda(R)$ and $\Lambda(S)$ are distinct but meet each other; R and S are not equivalent. Let x be a common point of $\mathrm{bd}\Lambda(R)$ and $\mathrm{bd}\Lambda(S)$, and $B(x)$ a tiny-ball neighborhood of x. By Lemma 7.3, $x \in M_h^o$ and so $x \in B(x)^o$. Let T be a crescent equivalent to R containing x, and T' that equivalent to S containing x. Then $T \cap B(x)$ is the closure of a component A of $B(x) - P$ for a totally geodesic $(n-1)$-ball P in $B(p)$ with boundary in $\mathrm{bd}B(x)$ by Lemma 3.13. Moreover, $\nu_T \cap B(x) = P$ and $T^o \cap B(x) = A$, and A is a subset of $\mathrm{int}\Lambda(R)$. Let B denote $B(x)$ removed with A and P. Similarly, $T' \cap B(x)$ is the closure of a component A' of $B(x) - P'$ for a totally geodesic $(n-1)$-ball P', $P' = \nu_{T'} \cap B(x)$ with boundary in $\mathrm{bd}B(x)$, and A' is a subset of T'^o in $\mathrm{int}\Lambda(S)$. Since we have

$$T^o \subset \mathrm{int}\Lambda(R), T' \subset \Lambda(S),$$

the sets $T' \cap B(x)$ and $T^o \cap B(x)$ are disjoint. Since P and P' contains x, it follows that $P = P'$ and $B = A'$; that is, P and P' are tangential. (We have that $P = P' = \nu_S \cap B(x) = \nu_{S'} \cap B(x)$.)

Since B is a subset of $\mathrm{int}\Lambda(S)$, B contains no point of $\Lambda(R)$ by Lemma 7.4, and similarly A contains no point of $\Lambda(S)$. Thus, $\Lambda(R) \cap B(x)$ is a subset of the closure of A, and $\Lambda(S) \cap B(x)$ is that of B. Since $A \subset \mathrm{int}\Lambda(R)$ and $B \subset \mathrm{int}\Lambda(S)$, it follows that

$$\begin{aligned} A &= \mathrm{int}\Lambda(R) \cap B(x), & B &= \mathrm{int}\Lambda(S) \cap B(x), \\ P \cup A &= \Lambda(R) \cap B(x), & P \cup B &= \Lambda(S) \cap B(x), \\ P &= \mathrm{bd}\Lambda(R) \cap B(x) & = \; & \mathrm{bd}\Lambda(S) \cap B(x). \end{aligned} \tag{7.2}$$

Hence, we have $P = \mathrm{bd}\Lambda(R) \cap \mathrm{bd}\Lambda(S) \cap B(x)$ and P is a totally geodesic $(n-1)$-ball with boundary in $\mathrm{bd}B(x)$ and our point x belongs to P^o, to begin with. Since this holds for an arbitrary choice of a common point x of $\mathrm{bd}\Lambda(R)$ and $\mathrm{bd}\Lambda(S)$, a tiny ball $B(x)$ of x, it follows that $\mathrm{bd}\Lambda(R) \cap \mathrm{bd}\Lambda(S) \cap M_h$ is an imbedded totally geodesic open $(n-1)$-submanifold in M_h^o. It is properly imbedded since $B(x) \cap \mathrm{bd}\Lambda(R) \cap \mathrm{bd}\Lambda(S)$ is compact for every choice of $B(x)$.

The above paragraph also shows that $\mathrm{bd}\Lambda(R) \cap \mathrm{bd}\Lambda(S) \cap M_h$ is an open and closed subset of $\mathrm{bd}\Lambda(R) \cap M_h$. Therefore, for components B of $\mathrm{bd}\Lambda(R) \cap M_h$ and B' of $\mathrm{bd}\Lambda(S) \cap M_h$ where $R \not\sim S$, either we have $B = B'$ or B and B' are disjoint. If $B = B'$, the above paragraph shows that B is a properly imbedded totally geodesic $(n-1)$-submanifold of M_h.

We say that a component of $\mathrm{bd}\Lambda(R) \cap M_h$ is *copied* if it equals a component of $\mathrm{bd}\Lambda(S) \cap M_h$ for some n-crescent S not equivalent to R. Let c_R be the union of all copied components of $\mathrm{bd}\Lambda(R) \cap M_h$.

Lemma 7.5. — *Each component of c_R is a properly imbedded totally geodesic $(n-1)$-manifold, and equals a component of $\nu_T \cap M_h$ for fixed T, $T \sim R$ and that of $\nu_{T'} \cap M_h$ for fixed T', $T' \sim S$, where S is not equivalent to R.*

Proof. — From above arguments, we see that given x in a component C of c_R, and a tiny ball $B(x)$ of x, there exists a totally geodesic $(n-1)$-ball P with $\delta P \subset \mathrm{bd}B(x)$ so that a component of $B(x) - P$ is included in T, $T \sim R$ and the other component in T' for T' equivalent to S but not equivalent to R.

Since P is connected, $P \subset C$. Let y be another point of C connected to x by a path γ in C, a subset of M_h. Then we can cover γ by a finitely many tiny balls. By induction on the number of tiny balls, we see that y belongs to $\nu_T \cap M_h$ and $\nu_{T'} \cap M_h$ for fixed T and T'. $\square$

Let A denote $\bigcup_{R\in\mathcal{B}} c_R$ where $\mathcal{B}$ denotes the set of representatives of the equivalence classes of bihedral n-crescents in $\check{M}_h$. A is said to be the *pre-two-faced submanifold arising from bihedral n-crescents*. A is a union of path-components that are totally geodesic $(n-1)$-manifolds closed in M_h^o.

Proposition 7.6. — *Suppose that A is not empty. Then A is a properly imbedded submanifold of M_h and $p|A$ is a covering map onto a closed totally geodesic imbedded $(n-1)$-dimensional submanifold in M^o.*

Proof. — We follow the argument in Chapter 6 somewhat repetitively. Every pair of two components a of c_R and b of c_S for n-crescents R and S where $R, S \in \mathcal{B}$, are either disjoint or identical. Hence, A is a union of disjoint closed path-components that are some components of c_R for $R \in \mathcal{B}$. This is proved exactly as in Chapter 6.

Second, given a tiny ball $B(x)$ of a point x of M_h, no more than one path-component of A may intersect $\mathrm{int}B(x)$. Let a be a component of c_R intersecting $\mathrm{int}B(x)$. By Lemma 7.5, a is a component of $\nu_S \cap M_h$ for $S \sim R$ and that of $\nu_T \cap M_h$ for T not equivalent to S. Furthermore, $\nu_S \cap B(x)$ is a compact convex $(n-1)$-ball with boundary in $\mathrm{bd}B(x)$. Since it is connected, $a \cap B(x) = \nu_S \cap B(x)$, and $B(x) \cap S$ is the closure of a component C_1 of $B(x) - (a \cap B(x))$. Similarly, $a \cap B(x) = \nu_T \cap B(x)$, and $B(x) \cap T$ is the closure of the other component C_2 of $B(x) - (a \cap B(x))$ for an n-crescent T not equivalent to R. Since S and T do not overlap, it follows that C_1 and C_2 are the two disjoint components of $B(x) - (a \cap B(x))$.

Suppose that b is a component of c_U for $U \in \mathcal{B}$ intersecting $\mathrm{int}B(x)$ also. By Lemma 7.5, b is a component of $\nu_{T'} \cap M_h$ for $T' \sim U$. If the $(n-1)$-ball $b \cap B(x)$ intersects C_1 or C_2, then $U \sim S$ or $U \sim T$ and $\Lambda(U) = \Lambda(R)$ or $\Lambda(U) = \Lambda(T)$ by Lemma 7.4, implying that $a = b$. If we have $b \cap B(x) \subset a \cap B(x)$, then T' overlaps with at least one of S or T, and $a = b$ as above.

Since given a tiny ball $B(x)$ no more than one distinct path-component of A may intersect $\mathrm{int}B(x)$, A is a properly imbedded closed submanifold of M_h^o. The rest of the proof of proposition is the same as that of Proposition 6.4. $\square$

Definition 7.7. — $p(A)$ for the union A of all copied components of $\Lambda(R)$ for bihedral n-crescents R in $\check{M}_h$ is said to be the *two-faced submanifold* of M *arising from bihedral n-crescents* (or *type II*).

Each component of $p(A)$ is covered by a component of A, i.e., a component of $\nu_R \cap M_h$ for some bihedral n-crescent R. Hence, each component of $p(A)$ is covered by open domains in $\mathbf{R}^n$ as in Chapter 6 and as we said in the introduction.

We end with the following observation:

Proposition 7.8. — *Suppose $\check{M}_h$ includes no hemispheric n-crescents and $A = \bigcup_{R\in\mathcal{B}} c_R$. Then A is disjoint from R^o for each n-crescent R.*

Proof. — The proof is same as that of Proposition 6.6. $\square$

Example 7.9. — Finally, we give an example in dimension 2. Let ϑ be the projective automorphism of $\mathbf{S}^2$ induced by the diagonal matrix with entries 2, 1, and 1/2. Then ϑ has fixed points $[\pm 1, 0, 0]$, $[0, \pm 1, 0]$, and $[0, 0, \pm 1]$ corresponding to eigenvalues $2, 1, 1/2$. Given three points x, y, z of $\mathbf{S}^2$, we let $\overline{xyz}$ denote the segment with endpoints x and z passing through y if there exists such a segment. If x and y are not antipodal, then let $\overline{xy}$ denote the unique minor segment with endpoints x and y. We look at the closed lune B_1 bounded by $\overline{[0,0,1][1,0,0][0,0,-1]}$ and $\overline{[0,0,1][0,1,0][0,0,-1]}$, which are to be denoted by l_1 and l_2 respectively, and the closed lune B_2 bounded by $\overline{[1,0,0][0,-1,0][-1,0,0]}$ and $\overline{[1,0,0][0,0,1][-1,0,0]}$, which are denoted by l_3 and l_4 respectively.

We consider the domain U given by $U = B_1^o \cup B_2^o \cup l_1^o \cup l_4^o - \{[1,0,0],[0,0,1]\}$. Since there exists a compact fundamental domain of the action of $\langle\vartheta\rangle$, $U/\langle\vartheta\rangle$ is a compact annulus A with totally geodesic boundary. U is the holonomy cover of A. The Kuiper completion of U can be identified with $B_1 \cup B_2$. It is easy to see that B_1 is a 2-crescent with $\alpha_{B_1} = l_2^o$ and $\nu_{B_1} = l_1$ and B_2 one with $\alpha_{B_2} = l_3^o$ and $\nu_{B_2} = l_4$. Also, any other crescent is a subset of B_1 or B_2. Hence $\Lambda(B_1) = B_1$ and $\Lambda(B_2) = B_2$ and the pre-two-faced submanifold L equals $\overline{[1,0,0][0,0,1]}^o$. L covers a simple closed curve in A given by $\overline{[1,0,0][0,0,1]}^o/\langle\vartheta\rangle$.

CHAPTER 8

THE PRESERVATION OF CRESCENTS AFTER DECOMPOSING AND SPLITTING

In this chapter, we consider somewhat technical questions: What become of the n-crescents in the completions of the holonomy cover of a submanifold in those of the holonomy cover of an ambient manifold? What happen to n-crescents in the completion of a manifold when we split the manifold along the two-faced manifolds. The answer will be that they are preserved in the best possible sense: Propositions 8.3, 8.7, and 8.10. In the process, we will define splitting manifolds precisely and show how to construct holonomy covers of split manifolds.

Also, from this chapter, covering spaces need not be connected, which only complicates the matter of identifying the fundamental groups with the deck transformation groups. Even for disconnected spaces we can define projective completions as long as immersions to $\mathbf{S}^n$, i.e., developing maps, are defined since we can always pull-back the metrics in that case.

For an alternative and more intuitive approach due to the referee to proving the materials here, see Remarks 8.4 and 8.12.

Let M be a real projective manifold with empty or totally geodesic boundary; let M_h be the holonomy cover of M with development pair $(\mathbf{dev}, h)$ and the group of deck transformations G_M; let $p : M_h \to M$ denote the covering map. Let N be a connected submanifold of M of codimension 0 with an induced real projective structure. Then $p^{-1}(N)$ is a codimension 0 submanifold of M_h. Choose a component A of $p^{-1}(N)$. Then A is a submanifold in M_h and $p|A$ covers N with the deck transformation group G_A equal to the group of deck transformations of M_h preserving A.

We claim that A is a holonomy cover of N with development pair $(\mathbf{dev}|A, h')$ where h' is a composition of the inclusion homomorphism and $h : G_M \to \mathrm{Aut}(\mathbf{S}^n)$. First, for each closed path in N which lifts to one in A obviously has a trivial holonomy (see Section 8.4 in [**28**]). Given a closed path in N with a trivial holonomy, it lifts to a closed path in M_h with a base point in A. Since A is a component of $p^{-1}(N)$, it follows that the closed path is in A. Therefore, A is the holonomy cover of N.

Lemma 8.1. — *Let A be a component of $p^{-1}(N)$ in M_h of a submanifold N of M. Then A is a holonomy cover of N with developing map $\mathbf{dev}|A$.* □

Let us discuss about the Kuiper completion of A. The path-metric on A is induced from the Riemannian metric on A induced from $\mathbf{S}^n$ by $\mathbf{dev}|A$. The completion of A with respect to the metric is denoted by $\check{A}$ and the set of ideal points A_∞; that is, $A_\infty = \check{A} - A$.

Note that $\check{A}$ may not necessarily equal the closure of A in $\check{M}_h$. A good example is the complement of the closure of the positive axis in $\mathbf{R}^2$ as A and $\mathbf{R}^2$ as M_h.

Let $i : A \to M_h$ be an inclusion map. Then i extends to a distance-decreasing map $\check{\imath} : \check{A} \to \mathrm{Cl}(A) \subset \check{M}_h$.

Lemma 8.2. —

(i) *$\check{\imath}^{-1}(M_{h,\infty})$ is a subset of A_∞.*

(ii) *If A is closed as a subset of M_h, then $\check{\imath}(A_\infty) \subset M_{h,\infty}$. Thus, in this case, $\check{\imath}^{-1}(M_{h,\infty}) = A_\infty$.*

(iii) *Let P be a submanifold in A with convex interior P^o. Then the closure P' of P in $\check{A}$ maps isometric to the closure P'' of P in $\check{M}_h$ under $\check{\imath}$. Here P' and P'' are tame.*

(iv) *$\check{\imath}$ maps $P' \cap A_\infty$ homeomorphic onto $P'' \cap M_{h,\infty}$.*

Proof. — (i) If x is a point of $\check{\imath}^{-1}(M_{h,\infty})$, then x does not belong to A since otherwise $\check{\imath}(x) = i(x) \in M_h$.

(ii) Suppose not. Then there exists a point x in M_h such that $x = \check{\imath}(y)$ for $y \in A_\infty$. There exists a sequence of points $y_i \in A$ with unique limit point y with respect to the path-metric $\mathbf{d}_A$ on A induced by μ, and hence, $y_i \to y$ with respect to $\mathbf{d}$ also. The sequence of points $i(y_i) = y_i \in A$ converges x since i is distance-decreasing. Therefore we obtain $y = x$ and $y \in A$, a contradiction.

(iii) Since $i|P^o$ is an isometry with respect to $\mathbf{d}_A$ and $\mathbf{d}$ on M_h, the third part follows.

(iv) By (i), the inverse image of $P'' \cap M_{h,\infty}$ under $\check{\imath}|P'$ is a subset of A_∞. By (ii), we see $\check{\imath}(P' \cap A_\infty) \subset P'' \cap M_{h,\infty}$. □

Suppose that A is a closed subset of M_h. Let R be an n-crescent in $\check{M}_h$, and consider a submanifold $R' = R \cap M_h$ with convex interior R^o. If R' is a subset of a submanifold A of M_h, then the above lemma shows that the closure R'' of R' in $\check{A}$ is isometric to R under $\check{\imath}$. By the above lemma, we obtain that R'' is also a crescent with $\alpha_{R''} = \check{\imath}^{-1}(\alpha_R)$, and $\nu_{R''} = \check{\imath}^{-1}(\nu_R)$. Moreover, if R is bihedral (resp. hemispheric), then R'' is bihedral (resp. hemispheric).

Conversely, let R be an n-crescent in $\check{A}$. By Lemma 8.2, $\check{\imath}|R : R \to \check{\imath}(R)$ is an imbedding, and the closure of $i(R \cap A)$ equals $\check{\imath}(R)$ and is a convex n-ball. By Lemma 8.2, $\check{\imath}(R)$ is an n-crescent, which is bihedral (resp. hemispheric) if R is bihedral (resp. hemispheric) since $\alpha_{\check{\imath}(R)} = \check{\imath}(\alpha_R)$ and $\nu_{\check{\imath}(R)} = \check{\imath}(\nu_R)$ hold.

Thus, we have proved.

Proposition 8.3. — *Let A be a submanifold of M_h closed as a subset of M_h. There exists a one to one correspondence of all bihedral n-crescents in $\check{A}$ and those in* $\mathrm{Cl}(A)$ *in $\check{M}_h$ given by $R \leftrightarrow R'$ for a bihedral n-crescent R in $\check{A}$ and R' one in* $\mathrm{Cl}(A)$ *if and only if $R^o = R'^o$. The same statement holds for hemispherical n-crescents.*

Remark 8.4. — An alternative proof suggested by the referee is as follows: If R is a n-crescent in $\check{A}$, then $R \cap M_h$ is an n-crescent set. By Remark 3.8, the closure of R in $\check{M}_h$ is a crescent. Conversely, given a crescent R in $\mathrm{Cl}(A)$, as $R \cap M_h$ is an n-crescent set in M_h, $R \cap M_h$ is an n-crescent set in A. Thus the closure of $R \cap M_h$ is an n-crescent in $\check{A}$.

We give a precise definition of splitting. Let N be a real projective n-manifold with a properly imbedded $(n-1)$-submanifold A. We take an open regular neighborhood N of A, which is an I-bundle over A. Let us enumerate components of A by $A_1, \dots, A_n, \dots$ and corresponding components of N by $N_1, \dots, N_n, \dots$ which are regular neighborhoods of $A_1, \dots, A_n, \dots$ respectively. (We do not require the number of components to be finite.)

For an i, N_i is an I-bundle over A_i. By parameterizing each fiber by a real line, N_i becomes a vector bundle over A with a flat linear connection. We see that there is a subgroup G_i of index at most two in $\pi_1(A_i)$ with trivial holonomy. The single or double cover $\tilde{N}_i$ of N_i corresponding to G_i is a product I-bundle over $\tilde{A}_i$ the cover of A_i corresponding to G_i, considered as a submanifold of $\tilde{N}_i$.

Since N_i is a product or twisted I-bundle over A_i, $N_i - A_i$ has one or two components. If $N_i - A_i$ has two components, then we take the closure of each components in N_i and take their disjoint union $\hat{N}_i$ which has a natural inclusion map $l_i : N_i - A_i \to \hat{N}_i$. If $N_i - A_i$ has one component, then take the double cover $\tilde{N}_i$ of N_i so that $\tilde{N}_i$ is now a product I-bundle over $\tilde{A}_i$. Then $N_i - A_i$ lifts and imbeds onto a component of $\tilde{N}_i - \tilde{A}_i$. We denote by $\hat{N}_i$ the closure of this component in $\tilde{N}_i$. There is a natural lift $l_i : N_i - A_i \to \hat{N}_i$, which is an imbedding. After we do this for each i, $i = 1, \dots, n, \dots$, we identify $N - A$ and the disjoint union $\coprod_{i=1}^{n} \hat{N}_i$ of all N_i by the maps l_i. When A is not empty, the resulting manifold M is said to be the *split manifold* obtained from N along A (this is just for terminological convenience).

We see that for each component of A, we get either two copies or a double cover of the component of A in the boundary of the split manifold M which are newly created by splitting. There is a natural quotient map $q : M \to N$ by identifying these new faces to A, i.e., $q|q^{-1}(A) : q^{-1}(A) \to A$ is a two-to-one covering map. Therefore, it is easy to see that M is compact if N is compact and M has totally geodesic boundary if A is totally geodesic.

Let N_h be a holonomy cover of N with a development pair $(\mathbf{dev}, h)$. We let G_N be the group of deck transformations of the covering map $p : N_h \to N$. If one splits N_h along the properly imbedded submanifold $p^{-1}(A)$, then it is easy to see that the split manifold M' covers the manifold M of N split along A with covering map p' obtained from extending p. However, M' may not be connected; for each component M_i of M, we choose a component M_i' of M' covering that component. M_i includes exactly one component P_i of $N - A$, and M_i' includes exactly one component P_i' of $N_h - p^{-1}(A)$ as a dense open subset. Thus, $\coprod_{i=1}^{n} P_i'$ covers $\coprod_{i=1}^{n} P_i$ and $\coprod_{i=1}^{n} M_i'$ covers M.

Remark 8.5. — The submanifold $p^{-1}(A)$ is orientable since great $(n-1)$-spheres in $\mathbf{S}^n$ are orientable and $\mathbf{dev}$ maps each components of $p^{-1}(A)$ into great $(n-1)$-spheres as immersions. Thus, there are no twisted I-bundle neighborhoods of components of $p^{-1}(A)$ as N_h is orientable also.

Let G_i be the subgroup of deck transformations of N_h acting on P_i', which is the group of deck transformations of the covering map $p|P_i' : P_i' \to P_i$. For each i, we define the homomorphism $h_i : G_i \to \mathrm{Aut}(\mathbf{S}^n)$ by $h \circ l_i$ where $l_i : G_i \to G_N$ is the homomorphism induced from the inclusion map. Since P_i' covers a component P_i of $N - A$, Lemma 8.1 shows that P_i' is a holonomy cover of that component with the development pair $(\mathbf{dev}'|P_i', h_i)$.

The developing map $\mathbf{dev}|P_i$ uniquely extends to a map from M_i' as an immersion for each i; we denote by $\mathbf{dev}' : \coprod_{i=1}^n M_i' \to \mathbf{S}^n$ the map obtained this way. It is easy to see that the action of G_i naturally extends to one on M_i' and becomes the group of deck transformations of the covering map $p|M_i' : M_i' \to M_i$. Since M_i' is obtained from P_i' by attaching boundary, it follows that M_i' is the holonomy cover of M_i with development pair $(\mathbf{dev}'|M_i', h_i)$. We say that the disjoint union $\coprod_{i=1}^n M_i'$ is a *holonomy cover* of $M = \coprod_{i=1}^n M_i$.

Suppose that there exists a nonempty pre-two-faced submanifold A of N_h arising from hemispheric n-crescents. Then we can split N by $p(A)$ to obtain M and N_h by A to obtain M', and M' covers M under the extension p' of the covering map $p : N_h \to N$.

We claim that the collection of hemispheric n-crescents in $\check{N}_h$ and the completion $\check{M}'$ of M' are in one to one correspondence. Let $q : M' \to N_h$ denote the natural quotient map identifying the newly created boundary components which restricts to the inclusion map $N_h - A \to N_h$. We denote by A' the set $q^{-1}(A)$, which are newly created boundary components of M'. Let $\check{M}'$ denote the projective completion of M' with the metric $\mathbf{d}$ extended from $N_h - A$. Then as q is distance-decreasing, q extends to a map $\check{q} : \check{M}' \to \check{N}_h$ which is one-to-one and onto on $M' - A' \to N_h - A$.

Lemma 8.6. — *$\check{q}$ maps A' to A, M' to N_h, and M'_∞ to $N_{h,\infty}$. (Which implies that $\check{q}^{-1}(A) = A'$, $\check{q}^{-1}(N_h) = M'$, and $\check{q}^{-1}(N_{h,\infty}) = M'_\infty$.)*

Proof. — The result of this lemma is essentially a consequence of the properness of the map q. We obviously have $\check{q}(A') = q(A') = A$ and $\check{q}(M') = q(M') = N_h$.

If a point x of $\check{M}'$ is mapped to that of A, then let γ be a path in $N_h - A$ ending at $\check{q}(x)$ in A. Then we may lift γ to a path γ' in $M' - A'$. $\check{q}(x)$ has a small compact neighborhood B in N_h where γ eventually lies in, and as the closure B' of a component of $B - A$ is compact, there exists a compact neighborhood B'' in M' mapping homeomorphic to B' under q and γ' eventually lies in B''. This means that x lies in B'' and hence in M'. As $q^{-1}(A) = A'$, x lies in A'. Thus, $\check{q}^{-1}(A) = A'$ and points of M'_∞ cannot map to a point of A.

Using a path-lifting argument, we may show that $\check{q}(M'_\infty)$ is a subset of $A \cup N_{h,\infty}$ as $\check{q}|M' - A' \to N_h - A$ is a homeomorphism and a $\mathbf{d}$-isometry. Hence, this and the above paragraph show that $\check{q}(M'_\infty) \subset N_{h,\infty}$. □

First, consider the case when $\check{N}_h$ includes a hemispheric n-crescent R. Since by Proposition 6.6, R^o is a subset of $N_h - A$, R^o is a subset of M'. The closure R' of

R^o in $\check{M}'$ is naturally an n-hemisphere as $\mathbf{dev}|R^o$ is an imbedding onto an open n-hemisphere in $\mathbf{S}^n$. As $\check{q}$ is a $\mathbf{d}$-isometry restricted to R^o, it follows that $\check{q}|R' : R' \to R$ is an imbedding.

Lemma 8.6 shows that $(\check{q}|R')^{-1}(\alpha_R)$ is a subset of M'_∞. Thus, R' includes an open $(n-1)$-hemisphere in $\delta R' \cap M'_\infty$, which shows that R' is a hemispheric n-crescent. ($\delta R'$ cannot belong to M'_∞ by Lemma 3.17.)

Now if an n-crescent R is given in $\check{M}'$, then we have $R^o \subset M' - A'$, and $\check{q}(R)$ is obviously an n-hemisphere as the closure R' of R^o in $\check{N}_h$ is an n-hemisphere and equals $\check{q}(R)$. Since $\check{q}(\alpha_R)$ is a subset of $N_{h,\infty}$ by Lemma 8.6, $\check{q}(R)$ is a hemispheric n-crescent.

Proposition 8.7. — *There exists a one-to-one correspondence between all hemispheric n-crescents in $\check{N}_h$ and those of $\check{M}'$ by the correspondence $R \leftrightarrow R'$ if and only if we have $R^o = R'^o$.*

Corollary 8.8. — *If $\check{N}_h$ includes a hemispheric n-crescent, then the Kuiper completion of the holonomy cover of at least one component of the split manifold M along the two-faced submanifold, also includes a hemispheric n-crescent.*

Proof. — Let M_i be the components of M and M'_i their holonomy cover as obtained earlier in this chapter; let P_i be the component of $N - p(A)$ in M_i and P'_i that of $N_h - A$ in M'_i so that P'_i covers P_i. We regard two components of $N_h - A$ to be equivalent if there exists a deck transformation of N_h mapping one to the other. Then P_i is a representative of an equivalence class $\mathcal{A}_i$. As the deck transformation group is transitive in an equivalence class $\mathcal{A}_i$, we see that given two elements P_i^a and P_i^b in $\mathcal{A}_i$, the components M_i^a and M_i^b of M' including them respectively are projectively isomorphic as the deck transformation sending P_i^a to P_i^b extends to a projective map $M_i^a \to M_i^b$, and hence to a quasi-isometry $\check{M}_i^a \to \check{M}_i^b$. Since real projective maps extending to quasi-isometries send hemispheric n-crescents to hemispheric n-crescents, if no $\check{M}'_i$ includes a hemispheric n-crescent, then it follows that $\check{M}'$ do not also. This contradicts Proposition 8.7. □

Proposition 8.9. — *If the Kuiper completions of holonomy cover of a submanifold of N or a split manifold of N by a properly-imbedded totally-geodesic closed submanifold in N includes hemispheric n-crescents, then so does $\check{N}_h$.*

Proof. — The first part follows from Proposition 8.3 and the second part follows from Lemma 8.6 as in the last part of the argument to prove Proposition 8.7. □

Now, we suppose that $\check{N}_h$ includes no hemispheric n-crescents R but includes some bihedral n-crescents. Let A be the pre-two-faced submanifold of N_h arising from bihedral n-crescents. We split N by $p(A)$ to obtain M and N_h by A to obtain M', and M' covers M under the extension p' of the covering map $p : N_h - A \to N - p(A)$.

By same reasonings as above, we obtain

Proposition 8.10. — *There exists a one-to-one correspondence between all bihedral n-crescents in $\check{N}_h$ and those of $\check{M}'$ by the correspondence $R \leftrightarrow R'$ if and only if $R^o = R'^o$.*

Corollary 8.11. — *If $\check{N}_h$ includes a bihedral n-crescent, then the projective completion of the holonomy cover of at least one component of the split manifold M along the two-faced submanifold, also includes a bihedral n-crescent.*

Remark 8.12. — An alternative proof of Propositions 8.7 and 8.10, we use the crescent sets (see Remark 3.8). As the interior of n-crescent sets are disjoint from pre-two-faced submanifolds, if we split along the submanifolds, we see that the boundary parts of n-crescents "double" along the pre-two-faced submanifolds, and hence, the crescent sets are preserved. This intuitive argument can be made into a proof quite easily.

Lastly, we will note the relationship between covering spaces and crescents with a proof sketched. This result will not be used but for the completeness sake we include it here.

Proposition 8.13. — *Let M^1 and M^2 be connected developing covers of M and M^1 covers M^2 by g. Then hemispheric (resp. bihedral) n-crescents in the Kuiper completion $\check{M}^1$ of M^1 correspond to hemispheric (resp. bihedral) n-crescents in the Kuiper completion $\check{M}^2$ of M^2 by the number of sheets $[\pi_1(M^1) : \pi_1(M^2)]$ to one.*

To begin a proof, we can choose developing maps $\mathbf{dev}^1$ and $\mathbf{dev}^2$ for M^1 and M^2 so that $\mathbf{dev}^2 = \mathbf{dev}^1 \circ g$ as in Proposition 3.4. Then we pull-back the metric $\mathbf{d}$ to $\mathbf{d}^1$ and $\mathbf{d}^2$ on M^1 and M^2 respectively, and obtain completed spaces $\check{M}^1$ and $\check{M}^2$. The covering map g extends to a distance-decreasing map $g' : \check{M}^1 \to \check{M}^2$.

We need to first show:

Lemma 8.14. — *g' maps the ideal set $\tilde{M}^1_\infty$ of $\check{M}^1$ into the ideal set $\tilde{M}^2_\infty$ of $\check{M}^2$; hence, $g'^{-1}(\tilde{M}^2_\infty) = \tilde{M}^1_\infty$.*

Proof. — The proof again uses a path-lifting argument and the properness of g. □

The next step is to show that given an n-crescent R in $\check{M}^1$, the image $g'(R)$ is an n-crescent in $\check{M}^2$. This follows by the fact that g' restricted to R is an imbedding and the above lemma 8.14. Given an n-crescent S in $\check{M}^2$, as $S \cap M^2$ is simply connected, there exists a set S' in M^1 mapping homeomorphic to $S \cap M^2$ by g. We can show easily that the closure of S' is an n-crescent. □

CHAPTER 9

THE CONSTRUCTION OF CONCAVE AFFINE MANIFOLDS

In this chapter we prove Theorem 1.1 using the previous three-sections, in a more or less straightforward manner. We will start with hemispheric crescent case and then the bihedral case. The proof of the bihedral case is entirely similar but will be spelled out

Definition 9.1. — A *concave affine manifold N of type I* is a real projective manifold such that its holonomy cover N_h is a subset of a hemispheric n-crescent in $\check{N}_h$. A *concave affine manifold N of type II* is a real projective manifold with concave or totally geodesic boundary so that N_h is a subset of $\Lambda(R)$ for a bihedral n-crescent R in $\check{N}_h$ and $\check{N}_h$ includes no hemispheric n-crescents. We allow N to have nonsmooth boundary that is concave.

It is easy to see that N is a concave affine manifold of type I if and only if $\check{N}_h$ equals a hemispheric n-crescent.

Given a real projective manifold N with a developing map $\mathbf{dev} : N_h \to \mathbf{S}^n$ and the holonomy homomorphism $h : \pi_1(N) \to \mathrm{Aut}(\mathbf{S}^n)$, suppose that $\mathbf{dev}(N_h)$ is a subset of an open n-hemisphere, i.e., an affine patch, and $h(\pi_1(N))$ acts on this hemisphere. Then obviously $h(\pi_1(N))$ restricts to affine transformations of the affine patch, and $(\mathbf{dev}, h)$ can be considered a development pair of an affine structure. Hence, N admits a natural affine structure.

If M is a concave affine manifold of type I, from the properties proved in the above Chapter 8, $\mathbf{dev}(\check{M}_h)$ equals an n-hemisphere H. Since the holonomy group acts on H, the interior M^o has a compatible affine structure. If M is one of the second type, then for each bihedral n-crescent R, $\mathbf{dev}$ maps $R \cap M_h$ into the interior of an n-hemisphere H (see Chapter 7). Hence, it follows that $\mathbf{dev}$ maps M_h into H^o. Since given a deck transformation ϑ, we have $\vartheta(\Lambda(R)) = \Lambda(\vartheta(R)) \supset M_h$, we obtain $\mathrm{int}\Lambda(\vartheta(R)) \cap \mathrm{int}\Lambda(R) \cap M_h \neq \emptyset$ and $R \sim \vartheta(R)$ by Lemma 7.4. This shows that $\Lambda(R) = \Lambda(\vartheta(R)) = \vartheta(\Lambda(R))$ and $\delta_\infty\Lambda(R) = \delta_\infty\Lambda(\vartheta(R)) = \vartheta(\delta_\infty\Lambda(R))$ for each deck transformation ϑ by equation 7.1. Since $\mathbf{dev}(\delta_\infty\Lambda(R))$ is a subset of a unique great sphere $\mathbf{S}^{n-1}$, it follows that $h(\vartheta)$ acts on $\mathbf{S}^{n-1}$ and since $\mathbf{dev}(M_h)$ lies in H^o, the holonomy group acts on H^o. Therefore, M has a compatible affine structure.

If M is a concave affine manifold of type I, then δM is totally geodesic since $M_h = R \cap M_h$ for an hemispheric n-crescent R and $\delta M_h = \nu_R \cap M_h$. If M is one of type II, then δM is concave, as we said in the definition above.

Let M be a compact real projective manifold with empty or totally geodesic boundary, and $p : M_h \to M$ the holonomy covering map with development pair $(\mathbf{dev}, h)$. For the purpose of the following lemma, we say that two n-crescents S and T, hemispheric or bihedral, to be *equivalent* if there exists a chain of n-crescents $T_1, T_2, \dots, T_n$ so that $S = T_1$ and $T = T_i$ and T_i and T_{i+1} overlap for each $i = 1, \dots, n-1$. We will use this definition in this chapter only.

Lemma 9.2. — *Let x_i be a sequence of points of M_h converging to a point x of M_h, and $x_i \in R_i$ for n-crescents R_i for each i. Then for any choice of an integer N, we have $R_i \sim R_j$ for infinitely many $i, j \geq N$. Furthermore, if each R_i is an n-hemisphere, then $R_i = R_j$ for infinitely many $i, j \geq N$. Finally x belongs to an n-crescent R for $R \sim R_i$ for infinitely many i.*

Proof. — Let $B(x)$ be a tiny ball of x. Assume $x_i \in \mathrm{int} B(x)$ for each i. We can choose a smaller n-crescent S_i in R_i so that x_i now belongs to ν_{S_i} with α_{S_i} included in α_{R_i} as R_i are geometrically "simple", i.e., a convex n-bihedron or an n-hemisphere.

Since $B(x)$ cannot be a subset of S_i, $S_i \cap B(x)$ is the closure of a component of $B(x) - a_i$ for $a_i = \nu_{S_i} \cap B(x)$ an $(n-1)$-ball with boundary in $\mathrm{bd} B(x)$. Let v_i be the outer-normal vector at x_i to ν_{S_i} for each i. Choose a subsequence i_j, with $i_1 = N$, of i so that the sequence v_{i_j} converges to a vector v at x. Corollary 3.16 shows that there exists an n-crescent R so that $\mathbf{dev}(R)$ is a limit of $\mathbf{dev}(S_{i_j})$, R contains x, and R and S_{i_j} include a fixed common n-ball $\mathcal{P}$ for j sufficiently large. Hence, S_{i_j} is equivalent to S_{i_k} for j, k sufficiently large. Since we have $R_{i_j} \sim S_{i_j}$ as S_{i_j} is a subset of R_{i_j}, we obtain $R_{i_j} \sim R_{i_k}$ for j, k sufficiently large.

If R_i are n-hemispheres, then Theorem 5.1 shows that $R_{i_j} = R_{i_k}$ for j, k sufficiently large. □

We begin the proof of the Main Theorem 1.1. Actually, what we will be proving is the following theorem, which together with Theorem 4.6 implies Theorem 1.1.

Theorem 9.3. — *Suppose that M is a compact real projective n-manifold with totally geodesic or empty boundary, and that M_h is not real projectively diffeomorphic to an open n-hemisphere or n-bihedron. Then the following statements hold*:

- *If $\check{M}_h$ includes a hemispheric n-crescent, then M includes a compact concave affine n-submanifold N of type I or M^o includes the two-faced $(n-1)$-submanifold arising from hemispheric n-crescent.*
- *If $\check{M}_h$ includes a bihedral n-crescent, then M includes a compact concave affine n-submanifold N of type II or M^o includes the two-faced $(n-1)$-submanifold arising from bihedral n-crescent.*

Converse statements also hold.

First, we consider the case when $\check{M}_h$ includes an n-crescent R that is an n-hemisphere. Suppose that there is no copied component of $\nu_T \cap M_h$ for every hemispheric n-crescent T. Recall from Chapter 6 that either $R = S$ or R and S are disjoint for every pair of hemispheric n-crescents R and S.

Let $x \in M_h$ and $B(x)$ the tiny ball of x. Then only finitely many distinct hemispheric n-crescents intersect a compact neighborhood of x in $\mathrm{int}B(x)$. Otherwise, there exists a sequence of points x_i converging to a point y of $\mathrm{int}B(x)$, where $x_i \in \mathrm{int}B(x)$ and $x_i \in R_i$ for mutually distinct hemispheric n-crescents R_i, but Lemma 9.2 contradicts this.

Consider $R \cap M_h$ for a hemispheric n-crescent R. Since R is a closed subset of $\check{M}_h$, $R \cap M_h$ is a closed subset of M_h. Let A be the set $\bigcup_{R \in \mathcal{H}} R \cap M_h$. Then A is a closed subset of M_h by above. Since $R \cap M_h$ is a submanifold for each n-crescent R, A is a submanifold of M_h, a closed subset. Since the union of all hemispheric n-crescents A is deck transformation group invariant, we have $p^{-1}(p(A)) = A$. Thus, $p|A$ is a covering map onto a compact submanifold N in M, and $p|R \cap M_h$ is a covering map onto a component of N for each hemispheric n-crescent R.

Since the components of A are locally finite in M_h, it follows that N has only finitely many components. Let K be a component of N. By Lemma 8.1, $R \cap M_h$ is a holonomy cover of K. Let $\check{K}$ be the projective completion of $R \cap M_h$. The closure of $R \cap M_h$ in $\check{K}$ is a hemispheric n-crescent identical with $\check{K}$ by Proposition 8.3. Hence, K is a concave affine manifold of type I.

If there is a copied component of $\nu_T \cap M_h$ for some hemispheric n-crescent T, Proposition 6.4 implies the Main theorem.

Now, we assume that $\check{M}_h$ includes only n-crescents that are n-bihedrons. Suppose that there is no copied component of $\mathrm{bd}\Lambda(T) \cap M_h$ for every bihedral n-crescents T, $T \in \mathcal{B}$. Then either $\Lambda(R) = \Lambda(S)$ or $\Lambda(R)$ and $\Lambda(S)$ are disjoint for n-crescents R and S, $R, S \in \mathcal{B}$, by the results of Chapter 7.

Using this fact and Lemma 9.2, we can show similarly to the proof for the hemispheric n-crescent case that $A = \bigcup_{R \in \mathcal{B}} \Lambda(R) \cap M_h$ is closed: $\Lambda(R) \cap M_h$ is a closed subset of M_h by Proposition 7.2. For each point x of M_h and a tiny ball $B(x)$ of x, there are only finitely many mutually distinct $\Lambda(R_i)$ intersecting a compact neighborhood of x in $\mathrm{int}B(x)$ for n-crescents R_i: Otherwise, we get a sequence x_i, $x_i \in \mathrm{int}B(x)$, converging to y, $y \in \mathrm{int}B(x)$, so that $x_i \in \Lambda(R_i)$ for n-crescents R_i with mutually distinct $\Lambda(R_i)$, i.e., R_i is not equivalent to R_j whenever $i \neq j$. Then $x_i \in S_i$ for an n-crescent S_i equivalent to R_i. Lemma 9.2 implies $S_i \sim S_j$ for infinitely many $i, j \geq N$. Since S_i is equivalent to R_i, this contradicts the fact that $\Lambda(R_i)$ are mutually distinct.

The subset A is a submanifold since each $\Lambda(R) \cap M_h$ is one for each R, $R \in \mathcal{B}$. Similarly to the hemisphere case, since $p^{-1}(p(A)) = A$, we obtain that $p|A$ is a covering map onto a compact submanifold N in M, and $p|\Lambda(R) \cap M_h$, $R \in \mathcal{B}$, is a covering map onto a component of N. N has finitely many components since the components of A are locally finite in M_h by the above paragraph.

Let K be the component of N that is the image of $\Lambda(R) \cap M_h$ for R, $R \in \mathcal{B}$. By Lemma 8.1, $\Lambda(R) \cap M_h$ is a holonomy cover of K. Let $\check{K}_h$ denote the projective completion of $\Lambda(R) \cap M_h$. For each crescent S, $S \sim R$, the closure S' of $S \cap M_h$ in $\check{K}_h$ is an n-crescent by Proposition 8.3. It follows that each point x of $\Lambda(R) \cap M_h$ is a point of a crescent S' in $\check{K}$ equivalent to the crescent R', the closure of $R \cap M_h$ in $\check{K}$. Therefore, by Lemma 7.3, N is a finite disjoint union of compact concave affine manifolds of type II.

When there are copied components of $\mathrm{bd}\Lambda(R) \cap M_h$ for some $R \in \mathcal{B}$, then Proposition 7.6 completes the proof of the Main theorem.

The converse part of the Main theorem follows by Proposition 8.3 since the Kuiper completions of concave submanifolds includes an n-crescents clearly. $\square$

CHAPTER 10

SPLITTING AND DECOMPOSING MANIFOLDS

In this chapter, we will prove Corollary 1.2. The basic tools are already covered in previous three chapters. As before, we study hemispheric case first.

Let M be a compact real projective n-manifold with empty or totally geodesic boundary. We will assume that M is not $(n-1)$-convex, and so M_h is not projectively diffeomorphic to an open n-bihedron or an open n-hemisphere, so that we can apply various results in Chapters 5 to 8, such as the intersection properties of hemispheric and bihedral n-crescents. We will carry out various decomposition of M in this chapter. Since in each of the following steps, the results are real projective manifolds with nonempty boundary if nontrivial decomposition had occurred, it follows that their holonomy covers are not projectively diffeomorphic to open n-bihedrons and open n-hemispheres. So our theory in Chapters 5 to 9 continues to be applicable.

We show a diagram of manifolds that we will be obtaining in the construction. The ladder in the first row is continued to the next one. Consider them as one continuous ladder.

$$\begin{array}{ccccccc}
M & \Rightarrow_{p(A_1)} & M^{\mathsf{s}} & \Rightarrow & N \coprod K & \Rightarrow_{p(A_2)} \\
\uparrow p & & \uparrow & & \uparrow & \\
M_h & \Rightarrow_{A_1} & M^{\mathsf{s}}_h & \Rightarrow & N_h \coprod \bigcup_{R\in\mathcal{H}}\{R\cap N_h\} & \Rightarrow_{A_2}
\end{array}$$

$$\begin{array}{ccc}
N^{\mathsf{s}} \coprod K & \Rightarrow & S \coprod T \coprod K \\
\uparrow & & \uparrow \\
N^{\mathsf{s}}_h \coprod \{\bigcup_{R\in\mathcal{H}} R\cap N_h\} & \Rightarrow & S_h \coprod \{\bigcup_{R\in\mathcal{B}} \Lambda(R)\cap N^{\mathsf{s}}_h\} \coprod \{\bigcup_{R\in\mathcal{H}} R\cap N_h\}
\end{array} \tag{10.1}$$

where the notation $\Rightarrow_A$ means to split along a submanifold A if A is compact and means to split and take appropriate components to obtain a holonomy cover if A is noncompact, $\Rightarrow$ means to decompose and to take appropriate components, $\coprod$ means a disjoint union and other symbols will be explained as we go along. When any of A_1, K, A_2, T is empty, then the operation of splitting or decomposition does not take place and the next manifolds are identical with the previous ones. For convenience, we will assume that all of them are not empty in the proof.

To begin, suppose that M is not $(n-1)$-convex, and we will now be decomposing M into various canonical pieces. Let $p : M_h \to M$ denote the holonomy covering map with development pair $(\mathbf{dev}, h)$. (Note that the covering maps in the spaces constructed below will be all denoted by p. Since the domains of definition are different this causes no confusion.)

Since M is not $(n-1)$-convex, $\check{M}_h$ includes an n-crescent (see Theorem 4.6). By Theorem 9.3, M has a two-faced $(n-1)$-manifold S, or M includes a concave affine manifold.

Suppose that $\check{M}_h$ has a hemispheric n-crescent, and that A_1 is a pre-two-faced submanifold arising from hemispheric n-crescents. (As before A_1 is two-sided.) Let M^{s} denote the result of the splitting of M along $p(A_1)$, and M' that of M_h along A_1, and A_1' the boundary of M' corresponding to A_1, "newly created from splitting." We know from Chapter 7 that there exists a holonomy cover M^{s}_h of M^{s} that is a disjoint union of suitable components of M'. This completes the construction of the first column of arrows in equation 10.1.

We now show that M^{s} now has no two-faced submanifold of type I. Let $\check{M}'$ denote the projective completion of M'. Suppose that two hemispheric n-crescents R and S in $\check{M}'$ meet at a common component C of $\nu_R \cap M'$ and $\nu_S \cap M'$ and that R and S are not equivalent. Proposition 6.4 applied to M' shows that $C \subset M'^o$; in particular, C is disjoint from A_1'.

Recall the map $\hat{q}: \check{M}' \rightarrow \check{M}_h$ extending the quotient map $q: M' \rightarrow M_h$ identifying newly created split faces in M'. There exist hemispheric n-crescents $\hat{q}(R)$ and $\hat{q}(S)$ in $\check{M}_h$ with same interior as R^o and S^o included in $M_h - A_1$ by Proposition 8.7.

Since $\nu_R \cap \nu_S \cap M'$ belongs to $M' - A_1' = M_h - A_1$, it follows that $q(\nu_R)$ and $q(\nu_S)$ meet in $M_h - A_1$. Since obviously $q(\nu_R) \subset \hat{q}(R)$ and $q(\nu_S) \subset \hat{q}(S)$, we have that $\hat{q}(R)$ and $\hat{q}(S)$ meet in $M_h - A_1$. However, since $\hat{q}(R)$ and $\hat{q}(S)$ are hemispheric n-crescents in $\check{M}_h$, $q(C)$ is a subset of the pre-two-faced submanifold A_1, which is a contradiction. Therefore, we have either $R = S$ or $R \cap S = \emptyset$ for n-crescents R and S in $\check{M}'$. Finally, since the completion $\check{M}^{\mathsf{s}}_h$ of the holonomy cover M^{s}_h is a subset of $\check{M}'$, we also have $R = S$ or $R \cap S = \emptyset$ for n-crescents R and S in $\check{M}^{\mathsf{s}}_h$.

The above shows that M^{s} has no two-faced submanifold arising from hemispheric n-crescents. Let $\mathcal{H}$ denote the set of all hemispheric n-crescents in $\check{M}^{\mathsf{s}}_h$. As in Chapter 9, $p|\bigcup_{R\in\mathcal{H}} R \cap M^{\mathsf{s}}_h$ is a covering map to the finite disjoint union K of compact concave affine manifolds of type I. Since any two hemispheric n-crescents are equal or disjoint, it is easy to see that $p^{-1}(K^o) = \bigcup_{R\in\mathcal{H}} R^o$. Then N, $N = M^{\mathsf{s}} - K^o$, is a real projective n-manifold with totally geodesic boundary; in fact, M^{s} decomposes into N and K along totally geodesic $(n-1)$-dimensional submanifold

$$p\Big(\bigcup_{R\in\mathcal{H}} \nu_R \cap M^{\mathsf{s}}_h\Big).$$

We see that $p^{-1}(N)$ equals $M^{\mathsf{s}}_h - p^{-1}(K^o)$, and so $M^{\mathsf{s}}_h - p^{-1}(K^o)$ covers N. As we saw in Chapter 8, we may choose a component L_i' of $M^{\mathsf{s}}_h - p^{-1}(K^o)$ for each component L_i of N covering L_i as a holonomy cover with the developing map $\mathbf{dev}|L_i'$ and a holonomy homomorphism as described there; $\coprod_{i=1}^n L_i'$ becomes a holonomy cover N_h of N by Lemma 8.1.

We will show that $\check{N}_h$ includes no hemispheric n-crescent, which implies that N includes no concave affine manifold of type I by the converse portion of Theorem 9.3. This completes the construction of the second column of arrows of equation 10.1.

The Kuiper completion of L'_i is denoted by $\check{L}'_i$. The Kuiper completion $\check{N}_h$ of N_h equals the disjoint union of $\check{L}'_i$. Since the inclusion map $i : L'_i \to M^{\mathsf{s}}_h$ is distance-decreasing, it extends to $\check{\imath} : \check{N}_h \to \check{M}^{\mathsf{s}}_h$. If $\check{N}_h$ includes any hemispheric n-crescent R, then $\check{\imath}(R)$ is a hemispheric n-crescent by Proposition 8.3. Since $\check{\imath}(R) \cap M_h$ is a subset of $p^{-1}(K)$ by the construction of K, and $R^o \subset \check{\imath}(R) \cap M^{\mathsf{s}}_h$, it follows that $R^o \subset p^{-1}(K)$. On the other hand, since $N_h \cap p^{-1}(K)$ equals $p^{-1}(\mathrm{bd} K)$, it follows that $N_h \cap p^{-1}(K)$ includes no open subset of M^{s}_h. Since $R^o \subset N_h$, this is a contradiction, Therefore, $\check{N}_h$ includes no hemispheric n-crescent.

We see after this stage that the completions of the covers of the subsequently constructed manifolds include no hemispheric n-crescents as the splitting and taking submanifolds do not affect this fact by Proposition 8.9.

Now we go to the second stage of the construction. Suppose that $\check{N}_h$ includes bihedral n-crescents and A_2 is the two-faced $(n-1)$-submanifold arising from bihedral n-crescents. Then we obtain the splitting N^{s} of N along $p(A_2)$.

We split N_h along A_2 to obtain $N^{\mathsf{s}\prime}$. Then the holonomy cover N^{s}_h of N^{s} is a disjoint union of components of $N^{\mathsf{s}\prime}$ chosen for each component of N^{s}. Let $\check{N}^{\mathsf{s}}_h$ denote the completion.

The reasoning using Proposition 8.10 as in the eighth paragraph above shows that $\Lambda(R) = \Lambda(S)$ or $\Lambda(R) \cap \Lambda(S) = \emptyset$ for every pair of bihedral n-crescents R and S in $\check{N}^{\mathsf{s}}_h$. Theorem 9.3 shows that N^{s} includes the finite disjoint union T of concave affine manifolds of type II with the covering map

$$p|\bigcup_{R\in\mathcal{B}} \Lambda(R) \cap N^{\mathsf{s}}_h : \bigcup_{R\in\mathcal{B}} \Lambda(R) \cap N^{\mathsf{s}}_h \to T$$

where $\mathcal{B}$ denotes the set of representatives of the equivalence classes of bihedral n-crescents in $\check{N}^{\mathsf{s}}_h$. And we see that $N^{\mathsf{s}} - T^o$ is a real projective manifold with convex boundary while T has concave boundary. By letting $S = N^{\mathsf{s}} - T^o$, we see that N^{s} decomposes into S and T.

Each component J of T is a maximal compact concave affine manifold of type II. If not, then J is a proper submanifold of a compact concave affine manifold J' of type II in N^{s}. A component of $p^{-1}(J)$ is a proper subset of a component of $p^{-1}(J')$. By Proposition 8.3, $p^{-1}(J')$ is a subset of $\bigcup_{R\in\mathcal{B}} \Lambda(R) \cap N^{\mathsf{s}}_h$. This is absurd.

For each component S_i of S, we choose a component S'_i of $N^{\mathsf{s}}_h - p^{-1}(T^o)$. Then $\coprod S'_i$ is a holonomy cover S_h of S. The projective completion $\check{S}_h$ equals the disjoint union $\coprod \check{S}'_i$. As in the sixth paragraph above, we can show that $\check{S}_h$ includes no bihedral n-crescent using Proposition 8.3. By the converse part of Theorem 9.3, we see that $\check{S}_h$ includes no compact concave affine manifold.

If S is not $(n-1)$-convex, then $\check{S}_h$ includes an n-crescent since the proof of Theorem 4.6 easily generalizes to the case when the real projective manifold M has convex boundary instead of totally geodesic one or empty one. Since $\check{S}_h$ does not include a hemispheric or bihedral n-crescent, it follows that S is $(n-1)$-convex.

Now, we will show that the decomposition of Corollary 1.2 is canonical. First, the two-faced submanifolds A_1 and A_2 are canonically defined. Now let $M = M^{\mathrm{s}}$ and $N = N^{\mathrm{s}}$ for convenience. First, suppose that M decomposes into N' and K' where K' is a submanifold whose components are compact concave affine manifolds of type I and N' is the closure of $M - K'$ and N' includes no compact concave affine manifold of type I. We will show that $N' = N$ and $K' = K$.

Let K'_i, $i = 1, \dots, n$, be the components of K', $K'_{i,h}$ their respective holonomy cover, and $\check{K}'_{i,h}$ the projective completions, which equals a hemispheric n-crescent R_i. We claim that $p^{-1}(K')$ equals $\bigcup_{R \in \mathcal{H}} R \cap M_h$ where $\mathcal{H}$ is the set of all n-crescents in $\check{M}_h$.

Each component K_i^j of $p^{-1}(K'_i)$ is a holonomy cover of K'_i (see Chapter 8). Let l_i^j denote the lift of the covering map $K'_{i,h} \to K'_i$ to K_i^j which is a homeomorphism. $\mathbf{dev} \circ l_i^j$ is a developing map for $K'_{i,h}$ as it is a real projective map (see Ratcliff [**28**]). We may put a metric $\mathbf{d}$ on $K'_{i,h}$ induced from $\mathbf{d}$ on $\mathbf{S}^n$, a quasi-isometric to any such choice of metric, using developing maps. Thus, we may identify K_i^j with $K'_{i,h}$ and their completions respectively for a moment.

From the definition of concave affine manifolds of type I, the completion of $K'_{i,h}$ equals a hemispheric n-crescent R_i, and $K'_{i,h} = R_i \cap K'_{i,h}$. Proposition 8.3 shows that there exists a hemispheric n-crescent R'_i in $\check{M}_h$ with identical interior as that of R_i, and clearly R'_i includes K_i^j in $\check{M}_h$ so that $K_i^j = R'_i \cap M_h$. Since this is true for any component K_i^j, we have that $p^{-1}(K')$ is a disjoint union of hemispheric n-crescents intersected with M_h and a subset of $\bigcup_{R \in \mathcal{H}} R \cap M_h$.

Suppose that there exists a hemispheric n-crescent R in $\check{M}_h$ so that $R \cap M_h$ is not a subset of $p^{-1}(K')$. Suppose R meets $p^{-1}(K')$. Then R meets a hemispheric n-crescent S where $S \cap M_h \subset p^{-1}(K')$. If R and S overlap, then $R = S$, which is absurd. Thus, $R \cap M_h$ and $S \cap M_h$ may meet only at $\nu_R \cap M_h$ and $\nu_S \cap M_h$. Hence, $R \cap M_h$ is a subset of $p^{-1}(M - K'^o)$. Since each component of $p^{-1}(M - K'^o)$ is a holonomy cover of a component of $M - K'^o$, it follows that the completion of a holonomy cover of a component of $M - K'^o$ includes a hemispheric n-crescent by Proposition 8.3.

The part N' has no two-faced submanifold of type I since otherwise we easily see that such a submanifold becomes a two-faced submanifold of type I for M itself by Proposition 8.9. Since N' includes no compact concave affine manifold of type I, the converse part of Theorem 9.3 shows that $\check{N}'_h$ includes no hemispheric n-crescent where $\check{N}'_h$ is the completion of the holonomy cover N'_h of N'. As this is a contradiction, we have that $p^{-1}(K')$ equals $\bigcup_{R \in \mathcal{H}} R \cap M_h$. Therefore, we obtain $N' = N$ and $K' = K$.

Second, if N decomposes into S' and T' where T' is the finite union of a maximal compact concave affine manifold of type II, and S' includes no compact concave affine manifold of type II, then we claim that $S' = S$ and $T' = T$. As above, we show that $p^{-1}(T')$ is a disjoint union of sets of form $\Lambda(R) \cap N_h$ for a bihedral crescent R in $\check{N}_h$ using maximality. As above, the converse part of Theorem 9.3 shows that the completion of the holonomy cover of each component of S' does not include a set of from $\Lambda(R) \cap N_h$ for a bihedral n-crescent R. The rest of proof is the same as in the hemispheric case. $\square$

CHAPTER 11

LEFT-INVARIANT REAL PROJECTIVE STRUCTURES ON LIE GROUPS

Finally, we end with an application to affine Lie groups. Let G be a Lie group with a left-invariant real projective structure, which means that G has a real projective structure and the group of left-translations are projective automorphisms.

As G is a manifold with real projective structure, there is an associated developing map $\tilde{G} \to \mathbf{S}^n$ and a holonomy homomorphism $\pi_1(G) \to \mathrm{Aut}(\mathbf{S}^n)$. Let G_h be the holonomy cover with induced development pair $(\mathbf{dev}, h)$. Then G_h is also a Lie group with the induced real projective structure, which is clearly left-invariant. Moreover, given an element g of G_h, as $\mathbf{dev} \circ L_g$ is another developing map for the left-translation L_g by g, $\mathbf{dev} \circ L_g = h'(g) \circ \mathbf{dev}$ for an element $h'(g)$ of $\mathrm{Aut}(\mathbf{S}^n)$. We see easily that $h' : G_h \to \mathrm{Aut}(\mathbf{S}^n)$ is a homomorphism, which is still said to be a *holonomy homomorphism.*

As before if G is not $(n-1)$-convex, then G_h is not projectively diffeomorphic to an open n-bihedron or an open n-hemisphere.

Theorem 11.1. — *If G is not $(n-1)$-convex as a real projective manifold, then the projective completion $\check{G}_h$ of G_h includes an n-crescent B.*

Proof. — This is proved similarly to Theorem 4.6 by a pull-back argument. The reason is that the left-action of G_h on G_h is proper and hence given two compact sets K and K' of G_h, the set $\{g \in G_h | g(K) \cap K' \neq \emptyset\}$ is a compact subset of G_h. That is, all arguments of Chapter 4 go through by choosing an appropriate sequence $\{g_i\}$ of elements of G_h instead of deck transformations.

Obviously, if G_h includes a cocompact discrete subgroup, then this is a corollary of Theorem 4.6. But if not, this parallel argument is needed. $\square$

Suppose that from now on $\check{G}_h$ includes an n-crescent B. Since the action of G_h on G_h is transitive, $\check{G}_h$ equals the union of $g(B)$ for $g \in G_h$. We claim that $g(B) \sim g'(B)$ for every pair of g and g' in G_h: That is, there exists a chain of n-crescents B_i, $i = 1, \dots, k$, of same type as B so that $B_1 = g(B)$, B_i overlaps with B_{i+1} for each $i = 1, \dots, k-1$, and $B_k = g'(B)$: Let x be a point of B^o and $B(x)$ a tiny ball of x in B^o. We can choose a sequence $g_0, \dots, g_n$ with $g_0 = g$ and $g_n = g'$ where $g_{i+1}^{-1} g_i(x)$

belongs to $B(x)^o$. In other words, we require $g_{i+1}^{-1}g_i$ to be sufficiently close to the identity element. Then $g_i(B^o) \cap g_{i+1}(B^o) \neq \emptyset$ for each i. Hence $g(B) \sim g'(B)$ for any pair $g, g' \in G_h$.

If B is an n-hemisphere, then we claim that $g(B) = B$ for all g and $G_h = B \cap G_h$. The proof of this fact is identical to that of Theorem 5.1 but we have to use the following lemma instead of Lemma 5.2.

Lemma 11.2. — *Suppose that* $\mathbf{dev} : \check{G}_h \to \mathbf{S}^n$ *is an imbedding onto the union of two n-hemispheres H_1 and H_2 meeting each other on an n-bihedron or an n-hemisphere. Then $H_1 = H_2$, and G_h is projectively diffeomorphic to an open n-hemisphere.*

Proof. — If H_1 and H_2 are different, as in the proof of Lemma 5.2 we obtain two $(n-1)$-dimensional hemispheres O_1 and O_2 in G_h where a subgroup of index one or two in G_h acts on. Since the action of G_h is transitive, this is clearly absurd. □

So if B is an n-hemisphere, then we obtain $G_h = B \cap G_h$. Since G_h is boundaryless, δB must consist of ideal points, which contradicts the definition of n-crescents. Therefore, every n-crescent in $\check{G}_h$ is a bihedral n-crescent.

The above shows that $G_h \subset \Lambda(B)$ for a bihedral n-crescent B, G_h is a concave affine manifold of type II and hence so is G: To prove this, we need to show that two overlapping n-crescents intersect transversally as the proof for the Lie group case is slightly different. The transversality is proved entirely as in the proof of Theorem 5.4 using Lemma 11.2 instead of Lemma 5.2.

Let H be a Lie group acting transitively on a space X. It is well-known that for a Lie group L with left-invariant (H, X)-structure, the developing map is a covering map onto its image, an open subset (see Proposition 2.2 in Kim [**23**]). Thus $\mathbf{dev} : G_h \to \mathbf{S}^n$ is a covering map onto its image.

Recall that $\mathbf{dev}$ maps $\Lambda(B)^o$ into an open subset of an open hemisphere H, and $\delta_\infty \Lambda(B)$ is mapped into the boundary $\mathbf{S}^{n-1}$ of H. Each point of G_h belongs to S^o for an n-crescent S equivalent to R since the action of G_h on G_h is transitive (see above). Since each point of $\mathbf{dev}(G_h)$ belongs to the interior of an n-bihedron S with a side in $\mathbf{S}^{n-1}$, the complement of $\mathbf{dev}(G_h)$ is a closed convex subset of H. Thus, $\mathbf{dev}|G_h$ is a covering map onto the complement of a convex closed subset of $\mathbf{R}^n$. As $\tilde{G}$ covers G_h, we see that this completes the proof of Theorem 1.4.

An affine m-convexity for $1 \leq m < n$ is defined as follows: Let M be an affine n-manifold, and T an affine $(m+1)$-simplex in $\mathbf{R}^n$ with sides $F_1, F_2, \dots, F_{m+2}$. Then M is *affine m-convex* if every nondegenerate affine map $f : T^o \cup F_2 \cup \cdots \cup F_{m+2} \to M$ extends to one $T \to M$ (see [**16**] for more details).

If G has a left-invariant affine structure, then G has a compatible left-invariant real projective structure. It is easy to see that the $(n-1)$-convexity of G in the real projective sense is equivalent to the affine $(n-1)$-convexity of G.

By Theorem 1.4, G is either $(n-1)$-convex or G_h is a concave affine manifold of type II. As before, if G_h is a concave affine manifold of type II, the argument above shows that G_h is mapped by $\mathbf{dev}$ to the complement of a closed convex set in $\mathbf{R}^n$. This completes the proof of Corollary 1.5.

Finally, we easily see that the following theorem holds with the same proof as the Lie group case:

Theorem 11.3. — *Let M be a homogenous space on which a Lie group G acts transitively and properly. Suppose M has a G-invariant real projective structure. Then M is either $(n-1)$-convex, or M is concave affine of type II. Also, M is affine $(n-1)$-convex or M is concave affine of type II if M has a G-invariant affine structure.*

PART III

APPENDICES

APPENDIX A

TWO MISCELLANEOUS THEOREMS

We prove in this section the fact that 3-manifolds with homogeneous Riemannian geometric structures admit real projective structures, and Theorem 4.5 on the equivalent definitions of convex real projective manifolds. The proofs are a little sketchy here; however, they are elementary.

Recall that given a pair of a space X and a Lie group G acting on X, Klein defined (X, G)-geometry as the G-invariant properties on X. An (X, G)-structure on a manifold is given by a maximal atlas of charts to X with transition functions lying in G.

Given the product space $\mathbf{R}P^2 \times \mathbf{R}P^1$, the group $\mathrm{PGL}(3, \mathbf{R}) \times \mathrm{PGL}(2, \mathbf{R})$ acts on the space in the standard manner; i.e., $(g, h)(x, y) = (g(x), h(y))$ for $x \in \mathbf{R}P^2$, $y \in \mathbf{R}P^1$, $g \in \mathrm{PGL}(3, \mathbf{R})$, and $h \in \mathrm{PGL}(2, \mathbf{R})$. The geometry modeled on the pair is said to be the *product real projective geometry* and the geometric structure modeled on the geometry is said to be the *product real projective structure.*

The following theorem is proved essentially using Molnár's work [**26**]. (See also Thiel [**32**].)

Theorem A.1. — *Let M be a 3-manifold with Riemannian homogeneous structure. Then M admits a real projective structure or a product real projective structure.*

Proof. — The Euclidean, spherical, and hyperbolic geometries correspond to the pairs $(\mathbf{R}^n, O(n, \mathbf{R}) \cdot \mathbf{R}^n)$, $(\mathbf{S}^n, O(n+1, \mathbf{R}))$, and $(H^n, \mathrm{PSO}(1, n))$. Here, $O(n, \mathbf{R})$ denotes the group of orthogonal transformations of $\mathbf{R}^n$, and $O(n, \mathbf{R}) \cdot \mathbf{R}^n$ the orthogonal group extended by translations, i.e., the group of rigid motions of $\mathbf{R}^n$. H^n denotes the positive part of the conic in $\mathbf{R}^{n+1}$ given by $x_0^2 - x_1^2 - \cdots - x_n^o = 1$, and $\mathrm{PSO}(1, n)$ the group of linear transformations acting on H^n.

As we said above, $\mathbf{R}^n$ is an affine patch and the group of rigid motions are affine, and hence projective. H^n can be identified with an open ball in $\mathbf{R}P^n$, with $\mathrm{PSO}(1, n)$ identified with an obvious copy in $\mathrm{PGL}(n+1, \mathbf{R})$. Hence, these geometries can be considered as a pair of open subsets in $\mathbf{R}P^n$ or $\mathbf{S}^n$ and subgroups of projective automorphisms of the open subsets respectively. Hence, an (X, G)-atlas for each of these geometries (X, G) is a real projective atlas, and so the three-manifold with an (X, G)-structure admits real projective structures.

Using the notation in Scott's paper [**29**], the geometries **Sol**, **Nil**, and $\widetilde{\mathrm{SL}}(2, \mathbf{R})$ can be realized as pairs of open subsets of real projective space $\mathbf{R}P^3$ or real projective spheres $\mathbf{S}^3$ and subgroups of projective automorphism groups. This can be seen in Molnár [**26**] as he gives explicit domains and the group of projective automorphisms corresponding to the isometry group. Hence, 3-manifolds admitting these structures also admit real projective structures.

For $H^2 \times \mathbf{R}^1$, and $\mathbf{S}^2 \times \mathbf{R}^1$ geometries, as two-dimensional hyperbolic and spherical geometries are realized by projective models, we see easily that they have models subsets of $\mathbf{R}P^2 \times \mathbf{R}P^1$ with the automorphism groups subgroups of $\mathrm{PGL}(3, \mathbf{R}) \times \mathrm{PGL}(2, \mathbf{R})$. $\square$

Theorem A.2. — *Let M be a real projective n-manifold. The following are equivalent*:

1. *M is 1-convex.*
2. *M is convex.*
3. *M is real projectively isomorphic to a quotient of a convex domain in $\mathbf{S}^n$.*

Furthermore, M_h can be identified with $\tilde{M}$ if any of the above items is true.

Proof. — (1)→(2): Since M is 1-convex, $\tilde{M}$ is 1-convex. Any two points x and y in $\tilde{M}$ are connected by a chain of segments s_i, $i = 1, \dots, n$, of $\mathbf{d}$-length $< \pi$ with endpoints p_i and p_{i+1} so that $s_i \cap s_{i+1} = \{p_{i+1}\}$ exactly. This follows since any path may be covered by tiny balls which are convex. We will show that x and y are connected by a segment of $\mathbf{d}$-length $\leq \pi$.

Assume that x and y are connected by such a chain with n being a minimum. We can assume further that s_i are in a general position, i.e., s_i and s_{i+1} do not extend each other as an imbedded geodesic for each $i = 1, \dots, n-1$, which may be achieved by perturbing the points $p_2, \dots, p_n$, unless $n = 2$ and $s_1 \cup s_2$ form a segment of $\mathbf{d}$-length π; in which case, we are done since $s_1 \cup s_2$ is the segment we need. To show we can achieve this, we take a maximal sequence of segments which extend each other as geodesics. Suppose that $s_i, s_{i+1}, \dots, s_j$ form such a sequence for $j > i$. Then the total length of the segment will be less than $\pi|j - i|$. We divide the sequence into new segments of equal $\mathbf{d}$-length ($< \pi$) $s'_i, s'_{i+1}, \dots, s'_j$ where s'_k has new endpoints p'_k, p'_{k+1} for $i \leq k \leq j$ where $p'_i = p_i$ and $p'_{j+1} = p_{j+1}$. Then we may change p'_k for $k = i+1, \dots, j$ toward one-side of the segments by a small amount generically. Since p'_k, $k = i+1, \dots, j$, are in an open hemisphere (an affine patch) determined by the original geodesic $s_i \cup \dots \cup s_j$, we see that new segments $s''_i, s''_{i+1}, \dots, s''_j$ are in general position together with s_{i-1} and s_{j+1}. This would work unless $j - i = 2$ and the total $\mathbf{d}$-length equals π since in this case changing p'_{i+1} still preserves $s''_i \cup s''_{i+1}$ to be a segment of $\mathbf{d}$-length π. However, since $n \geq 3$, we may move p'_i or p'_{j+1} in some direction to put the segments into a general position.

Let us choose a chain s_i, $i = 1, \dots, n$, of segments with minimal number of segments, in general position. We assume that we are not in case when $n = 2$ and $s_1 \cup s_2$ forming a segment of $\mathbf{d}$-length π,

We show that the number of the segments equals one, which shows that $\tilde{M}$ is convex. If the number of the segments is not one, then we take s_1 and s_2 and parameterize each of them by projective maps $f_i : [0, 1] \to s_i$, $i = 1, 2$, so that $f_i(0) = p_2$. Then since a tiny ball $B(p_2)$ can be identified with a convex ball in an affine patch, it follows that for t sufficiently small there exists a nondegenerate real projective map $f_t : \triangle_t \to B(p_2)$ where $\triangle_t$ is a triangle in $\mathbf{R}^2$ with vertices $(0, 0), (t, 0)$, and $(0, t)$ and $f_t(0, 0) = p_2$, $f_t(s, 0) = f_1(s)$ and $f_t(0, s) = f_2(s)$ for $0 \leq s \leq t$.

We consider the subset A of $(0, 1]$ so that $f_t : \triangle_t \to \tilde{M}$ is defined. For $t \in A$, $f_t : \triangle_t \to \tilde{M}$ is always an imbedding since $\mathbf{dev} \circ f_t$ is a nondegenerate projective map $\triangle_t \subset \mathbf{S}^n \to \mathbf{S}^n$. Thus A is open in $(0, 1]$ since as $f(\triangle_t)$ is compact, there exists a convex neighborhood of it in $\tilde{M}$ where $\mathbf{dev}$ restricts to an imbedding.

We claim that A is closed by 1-convexity: we consider the union $K = \bigcup_{t \in A} f_t(\triangle_t)$. Then the closure of K in $\check{M}$ is a compact triangle in $\check{M}$ with two sides in s_1 and s_2. Since two sides of K and K^o are in $\tilde{M}$, K itself is in $\tilde{M}$ by 1-convexity. Hence A must equal $(0, 1]$ and there exists a segment s_1' of $\mathbf{d}$-length $< \pi$, namely $f_1(\overline{(1, 0)(0, 1)})$, connecting p_1 and p_3. This contradicts the minimality, and x and y are connected by a segment of $\mathbf{d}$-length $\leq \pi$.

(2)$\to$(3) As $\tilde{M}$ is convex, the closure of $\tilde{M}$ in $\check{M}$ is tame as we explained in Chapter 3. Thus $\check{M}$ is a tame set, $\mathbf{dev} : \check{M} \to \mathbf{S}^n$ is an imbedding onto a convex subset of $\mathbf{S}^n$, and $\mathbf{dev}|\tilde{M}$ is an imbedding onto a convex subset of $\mathbf{S}^n$. Since the equation $\mathbf{dev} \circ \vartheta = h(\vartheta) \circ \mathbf{dev}$ holds for each deck transformation ϑ of $\tilde{M}$, it follows that $\mathbf{dev}$ induces a real projective diffeomorphism $\tilde{M}/\pi_1(M) \to \mathbf{dev}(\tilde{M})/h(\pi_1(M))$.

(3)$\to$(1) Since $\tilde{M}$ can be identified with a convex domain in $\mathbf{S}^n$, $\tilde{M}$ is 1-convex from the definition of 1-convexity. $\square$

APPENDIX B

SHRINKING AND EXPANDING n-BALLS BY PROJECTIVE MAPS

Proposition B.1. — *Suppose we have a sequence of ϵ-$\mathbf{d}$-balls B_i in a real projective sphere $\mathbf{S}^n$ for some $n \geq 1$ and a fixed positive number ϵ and a sequence of projective maps φ_i. Assume the following*:

- *The sequence of $\mathbf{d}$-diameters of $\varphi_i(B_i)$ goes to zero.*
- *$\varphi_i(B_i)$ converges to a point, say p, in the Hausdorff sense.*
- *For a compact n-ball neighborhood L of p, $\varphi_i^{-1}(L)$ converges to a compact set L_∞.*

Then L_∞ is an n-hemisphere.

Recall that $\mathbf{R}^{n+1}$ has a standard Euclidean metric and $\mathbf{d}$ on $\mathbf{S}^n$ is obtained from it by considering $\mathbf{S}^n$ as the standard unit sphere in $\mathbf{R}^{n+1}$.

The Cartan decomposition of Lie groups states that a real reductive Lie group G can be written as KTK where K is a compact Lie group and T is a maximal real tori. Since $\mathrm{Aut}(\mathbf{S}^n)$ is isomorphic to $\mathrm{SL}_\pm(n+1, \mathbf{R})$, we see that $\mathrm{Aut}(\mathbf{S}^n)$ can be written as $O(n+1)D(n+1)O(n+1)$ where $O(n+1)$ is the orthogonal group acting on $\mathbf{S}^n$ as the group of isometries and $D(n+1)$ is the group of determinant 1 diagonal matrices with positive entries listed in decreasing order where $D(n+1)$ acts in $\mathbf{S}^n$ as a subgroup of $\mathrm{GL}(n+1, \mathbf{R})$ acting in the standard manner on $\mathbf{S}^n$. In other words, each element g of $\mathrm{Aut}(\mathbf{S}^n)$ can be written as $i(g)d(g)i'(g)$ where $i(g)$ and $i'(g)$ are isometries and $d(g) \in D(n+1)$ (see Carrière [**8**] and Choi [**16**], and also [**15**]).

We may write φ_i as $K_{1,i} \circ D_i \circ K_{2,i}$ where $K_{1,i}$ and $K_{2,i}$ are $\mathbf{d}$-isometries of $\mathbf{S}^n$ and D_i is a projective map in $\mathrm{Aut}(\mathbf{S}^n)$ represented by a diagonal matrix of determinant 1 with positive entries. More precisely, D_i has $2n+2$ fixed points $[\pm e_0], \dots, [\pm e_n]$, the equivalence classes of standard basis vectors $\pm e_0, \dots, \pm e_n$ of $\mathbf{R}^{n+1}$, and D_i has a matrix diagonal with respect to this basis; the diagonal entries λ_i, $i = 0, 1, \dots, n$, are positive and in decreasing order. Let $O_{[e_0]}$ denote the open n-hemisphere containing $[e_0]$ whose boundary is the great sphere $\mathbf{S}^{n-1}$ containing $[\pm e_j]$ for all j, $j \geq 1$, and $O_{[-e_0]}$ that containing $[-e_0]$ with the same boundary set.

Since $\varphi_i(B_i)$ converges to p, and $K_{1,i}$ is an isometry, the sequence of the $\mathbf{d}$-diameter of $D_i \circ K_{2,i}(B_i)$ goes to zero as $i \to \infty$. We may assume without loss of generality that $D_i(K_{2,i}(B_i))$ converges to a set consisting of a point by choosing a subsequence if necessary. By the following lemma B.2, $D_i(K_{2,i}(B_i))$ converges to one of the attractors $[e_0]$ and $[-e_0]$. We may assume without loss of generality that $D_i(K_{2,i}(B_i))$ converges to $[e_0]$.

Since L is an n-dimensional ball neighborhood of p, L includes a $\mathbf{d}$-ball $B_\delta(p)$ in $\mathbf{S}^n$ with center p with radius δ for some positive constant δ. There exists a positive integer N so that for $i > N$, we have

$$\varphi_i(p_i) \subset B_{\delta/2}(p)$$

for the $\mathbf{d}$-ball $B_{\delta/2}(p)$ of radius $\delta/2$ in $\mathbf{S}^n$. Letting $q_i = \varphi_i(p_i)$ for the $\mathbf{d}$-center p_i of the ball B_i, we see that $B_{\delta/2}(q_i)$ is a subset of L for $i > N$.

Since $K_{1,i}^{-1}(q_i) = D_i \circ K_{2,i}(p_i)$, the sequence $K_{1,i}^{-1}(q_i)$ converges to $[e_0]$ by the second paragraph above. There exists an integer N_1, $N_1 > N$, such that $K_{1,i}^{-1}(q_i)$ is of $\mathbf{d}$-distance less than $\delta/4$ from $[e_0]$ for $i > N_1$. Since $K_{1,i}^{-1}$ is a $\mathbf{d}$-isometry, $K_{1,i}^{-1}(B_{\delta/2}(q_i))$ includes the ball $B_{\delta/4}([e_0])$ for $i > N_1$. Hence $K_{1,i}^{-1}(L)$ includes $B_{\delta/4}([e_0])$ for $i > N_1$.

Since $[e_0]$ is an attractor under the action of the sequence $\{D_i\}$ by Lemma B.2, the images of $B_{\delta/4}([e_0])$ under D_i^{-1} eventually include any compact subset of $O_{[e_0]}$. Thus, $D_i^{-1}(B_{\delta/4}([e_0]))$ converges to $\mathrm{Cl}(O_1)$ in the Hausdorff metric, and up to a choice of a subsequence $K_{2,i}^{-1} \circ D_i^{-1}(B_{\delta/4}([e_0]))$ converges to an n-hemisphere. The equation

$$\begin{aligned} \varphi_i^{-1}(L) &= K_{2,i}^{-1} \circ D_i^{-1} \circ K_{1,i}^{-1}(L) \\ &\supset K_{2,i}^{-1} \circ D_i^{-1}(B_{\delta/4}([e_0])). \end{aligned} \tag{B.1}$$

shows that $\varphi_i^{-1}(L)$ converges to an n-hemisphere. □

The straightforward proof of the following lemma is left to the reader.

Lemma B.2. — *Let K_i be a sequence of ϵ-$\mathbf{d}$-balls in $\mathbf{S}^n$ and d_i a sequence of automorphisms of $\mathbf{S}^n$ that are represented by diagonal matrices of determinant 1 with positive entries for the standard basis with the first entry λ_i the maximum. Suppose $d_i(K_i)$ converges to the set consisting of a point y. Then there exists an integer N so that for $i > N$, the following statements hold:*

1. *$[e_0]$ and $[-e_0]$ are attracting fixed points of d_i.*
2. *y equals $[e_0]$ or $[-e_0]$.*
3. *The eigenvalue λ_i of d_i corresponding to e_0 and $-e_0$ is strictly larger than the eigenvalues corresponding to $\pm e_j$, $j = 1, \dots, n$.*
4. *$\lambda_i/\lambda_i' \to +\infty$ for the maximum eigenvalue λ_i' of d_i corresponding to $\pm e_j, j = 1, \dots, n$.*

FREQUENTLY USED SYMBOLS

A^o: the topological interior of a convex set A in $\langle A\rangle$.
$\langle A\rangle$: the unique minimal great sphere including a convex subset A of $\mathbf{S}^n$.
α_R: the union of all open $(n-1)$-hemispheres in the intersection of δR with the ideal set if R is a hemispheric n-crescent, or the interior of a side of R in the ideal set if R is a bihedral n-crescent.
$\mathrm{bd}A$: the topological boundary of the set A with respect to the obvious largest ambient space.
$\mathrm{Cl}(A)$: the topological closure of A in the obvious largest ambient space.
$\mathrm{int}A$: the topological interior of A in the obvious largest ambient space.
c_R: the union of copied components of a hemispheric n-crescent R.
δA: the manifold boundary of a manifold A.
$\delta_\infty\Lambda(R)$: the union of α_S for all bihedral n-crescents S equivalent to an n-crescent R.
$\Lambda(R)$: the union of all bihedral n-crescents equivalent to an n-crescent R.
M^o: the manifold interior of a manifold M.
$\check{M}$: the Kuiper completion of the universal cover of M.
$\tilde{M}$: the universal cover of a manifold M.
M_h: the holonomy cover of a real projective manifold M.
$\check{M}_h$: the Kuiper completion of a holonomy cover M_h.
$M_{h,\infty}$: the set of ideal points of the Kuiper completion of a holonomy cover M_h.
$\tilde{M}_\infty$: the set of ideal points of the Kuiper completion of a universal cover $\tilde{M}$.
ν_R: the complement of α_R in δR if R is a hemispheric n-crescent or the side of R not in the ideal set if R is a bihedral n-crescent.
∂A: the topological boundary of a convex subset A in $\langle A\rangle$ if A is a subset of $\mathbf{S}^n$, or the subset of $\mathrm{Cl}(A)$ corresponding to ∂B for the image B of A under the developing map if A is a tame subset of a Kuiper completion.
$\pi_1(M)$: the fundamental group of a manifold M or the deck transformation group.
$\mathbf{R}$: the real number field.
$\mathbf{R}P^n$: the n-dimensional real projective space.

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