3.5. The geometry of linear systems

Solutions for inhomogeneous systems.
Consistency
Geometric interpretations

Translated subspaces

- W is a subspace.
- $x\_0+W=\{v=x\_0+w|w \text{ is in } W\}$
- This is not a subspace in general but is called an affine subspace (linear manifold, flat).
- For example $x\_0+\text{span}\{v\_0,v\_1,\ldots,v\_s\}$ $=\{v=x\_0+c\_0v\_1+\ldots+c\_sv\_s\}$
- y=1 in R^2. $\{(x,1)|x \text{ in } R\}=(0,1)+\{(x,0)|x \text{ in } R\}$
- Ax+By+Cz=D in R^3 translated from Ax+By+Cz=0 since they are parallel.

The solution space of Ax=b and that of Ax=0

- W={x|Ax=b}, W_O={x|Ax=O}
- Let x be in W. Take one x_0 in W. Then x-x_0 is in W_O.
  - A(x-x_0)=Ax-Ax_0=b-b=0.
- Given an element x in W_O. x+x_0 is in W.
- A(x+x_0)=Ax+Ax_0=O+b=b.
- Thus, W=x_0+W_O.

Theorem 3.5.1 *If* $A\mathbf{x} = \mathbf{b}$ *is a consistent nonhomogeneous linear system, and if* W *is the solution space of the associated homogeneous system* $A\mathbf{x} = \mathbf{0}$, *then the solution set of* $A\mathbf{x} = \mathbf{b}$ *is the translated subspace* $\mathbf{x}_0 + W$, *where* $\mathbf{x}_0$ *is any solution of the nonhomogeneous system* $A\mathbf{x} = \mathbf{b}$ (*Figure* 3.5.1).

- W ={(x,y)|x+y=1} is obtained from
- W_0 ={(x,y)|x+y=0} adding (1,0) in W.
- W={(x,y,z)|Ax+By+Cz=D} is obtained from W_0={(x,y,z)|Ax+By+Cz=O} by a translation by (x_0,y_0,z_0) for any point of W.
- W ={(x,y,z)| x+y+z=1, x-y=0 }
  - ={(s+1/2,1/2,s)|s in R}
  - =

$$\begin{bmatrix} s+1/2 \\ 1/2 \\ s \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

  - Here (1/2,1/2,0) is in W and {s(1,0,1)} are solutions of the homogeneous system.

- Solution to Ax=b can be written as x=x_h+x_0 where x_0 is a particular solution and x_h is a homogeneous solution.

Theorem 3.5.2 *A general solution of a consistent linear system* $A\mathbf{x} = \mathbf{b}$ *can be obtained by adding a particular solution of* $A\mathbf{x} = \mathbf{b}$ *to a general solution of* $A\mathbf{x} = \mathbf{0}$.

Theorem 3.5.3 *If* A *is an* $m \times n$ *matrix, then the following statements are equivalent.*

(a) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(b) $A\mathbf{x} = \mathbf{b}$ *has at most one solution for every* $\mathbf{b}$ *in* R^m *(i.e., is inconsistent or has a unique solution).*

Theorem 3.5.4 *A nonhomogeneous linear system with more unknowns than equations is either inconsistent or has infinitely many solutions.*

Consistency of a linear equation.

- Ax=b can be written as x_1v_1+x_2v_2+…+x_nv_n=b.

Theorem 3.5.5 *A linear system* $A\mathbf{x} = \mathbf{b}$ *is consistent if and only if* $\mathbf{b}$ *is in the column space of* A.

- This can be used to tell whether a certain vector can be written as a linear combination of some other vectors
- Example 2.

Hyperplanes

- $a_1x_1+a_2x_2+\ldots+a_nx_n=b$ in R^n. (a_i not all zero)
- The set of points $(x_1,x_2,\ldots,x_n)$ satisfying the equation is said to be a hyperplane.
- $b=0$ if and only if the hyperplane passes O.
- We can rewrite $a.x=b$ where $a=(a_1,\ldots,a_n)$ and $x=(x_1,..,x_n)$.
- A hyperplane with normal a.
- $a.x=0$. An orthogonal complement of a.
- Example 3.

Geometric interpretations of solution spaces.

- $a_{11} x_1+a_{12} x_2+\ldots+a_{1n} x_n=b_1$
- $a_{21} x_1+a_{22} x_2+\ldots+a_{2n} x_n=b_2$
- …….
- $a_{m1} x_1+a_{m2} x_2+\ldots+a_{mn} x_n=b_m$

- This can be written: $a_1.x=0$, $a_2.x=0,\ldots,a_m.x=0$.

Theorem 3.5.6 *If A is an $m \times n$ matrix, then the solution space of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in R^n that are orthogonal to every row vector of A.*

- See Example 4

Look ahead

- The set of solutions of a system of linear equation can be solved by Gauss-Jordan method.
- The result is the set W of vectors of form
  $x_0+t_1v_1+\ldots+t_sv_s$
  where t_is are free variables.
- We show that $\{v_1,v_2,\ldots,v_n\}$ is linearly independent later.
- Thus $W = x_0+W_0$. W is an affine subspace of dimension s.

Ex. Set 3.5.

- 1-4 solving
- 5-8 linear combinations
- 7-10 span
- 11-20 orthogonality