# 1.2.Dot Products, Orthogonality

# lengths

- Length, norm, magnitude of a vector $v=(v_1,\ldots,v_n)$ is $||v|| = (v_1^2+v_2^2+\ldots+v_n^2)^{1/2}$.
- Examples $v=(1,1,\ldots,1)$ $||v||=n^{1/2}$.
- Unit vectors $u=v/||v||$ corresponds to directions.
- Standard unit vectors
  - $i=(1,0)$, $j=(0,1)$ in $R^2$
  - $i=(1,0,0)$, $j=(0,1,0)$, $k=(0,0,1)$ in $R^3$
  - $e_1=(1,0,..,0)$, $e_2=(0,1,\ldots,0)$, $\ldots,e_n=(0,0,\ldots,1)$ in $R^n$.

- $v=(v_1,v_2, \ldots,v_n)= v_1e_1 + v_2e_2 + \ldots +v_n e_n$.
  - This is a unique expression.
- Distances when given position vectors:
- P(x_1,x_2) = ||P_1P_2||=((x_2-x_1)$^2$+(y_2-y_1)$^2$)$^{1/2}$ in 2-space.
- In 3-space d(P_1,P_2) = ||P_2-P_1||=((x_2-x_1)$^2$+(y_2-y_1)$^2$+(z_2-z_1)$^2$)$^{1/2}$ in 2-space.
- In n-space u=(u_1,u_2,…,u_n), v=(v_1,v_2, …,v_n),

Then d(u,v) = ((u_1-v_1)$^2$+(u_2-v_2)$^2$+…+ (u_n-v_n)$^2$)$^{1/2}$.

- Theorem. Two position vectors u,v in $R^n$.
  - $d(u,v)\geq 0$, $d(u,v)=d(v,u)$, $d(u,v)=0$ if and only if $u=v$.
- Proof: Use the formula.

- We now introduce dot product. Given two vectors, the dot product gives you a real number.
- The dot product generalizes length and angle and is useful to compute many quantities. In fact, it is more fundamental than angles. Given $u=(u_1,u_2,\ldots,u_n)$, $v=(v_1,v_2,\ldots,v_n)$,

$$u\bullet v = u_1v_1+u_2v_2+\ldots+u_nv_n.$$

# Properties of dot products

- $||v||=(v\cdot v)^{1/2}$.
- Theorem 1.2.7
  - $u\cdot v=v\cdot u$, Symmetry
  - $u\cdot(v+w)= u\cdot v+u\cdot w$. distributivity
  - $k(u\cdot v)=(ku)\cdot v$ homogenous
  - $v\cdot v\geq 0$, and $v\cdot v=0$ if and only if $v=0$. positivity
- Theorem 1.2.8.
  - $0\cdot v=v\cdot 0=0$
  - $(u+v)\cdot w=u\cdot w+v\cdot w$
  - $u\cdot(v-w)=u\cdot v-u\cdot w$, $(u-v)\cdot w=u\cdot w-v\cdot w$
  - $k(u\cdot v)=u\cdot(kv)$

- Theorems 1.2.6,1.2.7 gives us a means to compute as one does with real numbers. (See board.)
- Theorem 1.2.8:u, v nonzero vectors in $R^2$, $R^3$. If θ is an angle between u and v, then

  $$\cos\theta = u\bullet v / ||u\bullet v|| \text{ or } \theta = \cos^{-1}(u\bullet v/||u\bullet v||).$$

- Proof: Use cosine law
  - $||v-u||^2 = ||u||^2 + ||v||^2 - 2||u||\,||v||\cos\theta$.
  - Now $||v-u||^2 = (v-u)\bullet(v-u) = (v-u)\bullet v - (v-u)\bullet u = v\bullet v - u\bullet v - v\bullet u + u\bullet u = ||v||^2 - 2u\bullet v + ||u||^2$.
  - $||v||^2 - 2u\bullet v + ||u||^2 = ||u||^2 + ||v||^2 - 2||u||\,||v||\cos\theta$.
  - We simplify to get above.

- We consider $\theta$ to be in $[0,\pi]$ interval.
- Orthogonality.
  - $u \bullet v = 0$ iff $\cos \theta = 0$ iff $\theta = \pi/2$.
  - Two nonzero vectors in 2- or 3-spaces are perpendicular if and only if their dot product is zero.
- See Example 5,6.(See board)
- Definition. We extend he above formula to hold for n-space as well.
- Thus two vectors in n-spaces are *orthogonal* if their dot product is zero. A nonempty set of vectors is said to be an orthogonal set if each pair of distinct vectors are orthogonal.
- Use perpendicular for nonzero-vectors.

- Zero vector 0 is orthorgonal to every vector in $R^n$. Actually, it is the only such vector in $R^n$.
- {(1,0,0),(0,1,0),(0,0,1)}
- Orthonormal set. Two vectors are orthonormal if they are orthogonal and have length 1. A set of vectors is *orthonormal* if every vector in the set has length 1 and each pair of vectors is orthogonal.

- Phythagoras theorem: If u and v are orthogonal vectors, then $||u+v||^2=||u||^2+||v||^2$.
- Proof: $||u+v||^2 =(u+v).(u+v)=||u||^2+||v||+2u.v = ||u||^2+||v||^2$.
- Cauchy-Swartz inequality
  - $(u.v)^2 \leq ||u||^2||v||^2$ or $|(u.v)|\leq ||u||||v||^2$
- Proof: If u=0 or v=0, then true.
  - (See board.)
- Triangle inequality: u,v,w vectors.
  - $||u+v|| \leq||u||+||v||$.
- Proof: $||u+v||^2=(u+v).(u+v)=$
- $||u||^2+2(u.v)+||v||^2 \leq ||u||^2+2|u.v|+||v||^2\leq$
- $||u||^2+2||u||||v||+||v||^2$

- Theorem 1.2.14. Parallelogram equation for vectors. $||u+v||^2+||u-v||^2 = 2(||u||^2+||v||^2)$.
- Proof: see board
- Triangle inequality: u,v,w vectors $d(u,v) \leq d(u,w)+d(w,v)$.
- Proof: see board.

# 1.3. Vector equations of lines and planes

# Lines

- General equation for lines in 2-space: $Ax+By=C$. (A,B not both zero)
- $Ax+By=0$ (passes origin)

- Another method (parametric equation): Let a line pass through $x_0$.
  If $x$ is a point on the line, then $x-x_0$ is always parallel to a fixed vector say $v$.
- Thus $x-x_0=tv$ for some real number $t$.
- $x = x_0+tv$. ($t$ is called a parameter)

- $(x,y)=(x_0,y_0)+t(a,b)$.
- $x=x_0+ta$, $y=y_0+tb$.

- In 3-space, $(x,y,z)=(x_0,y_0,z_0)+t(a,b,c)$. Thus, $x=x_0+ta$, $y=y_0+tb$, $z=z_0+tc$.

- Given two points $x_1$, $x_0$ in $R^2$ or $R^3$, we try to find a line through them.
  - The line is parallel to $x_1 - x_0$.
  - Thus $x= x_0+t(x_1-x_0)$ or $x= (1-t)x_0+tx_1$.
  - This is a *two-point vector equation*.
  - If t is in [0,1], then the point is in the segment with endpoints $x_0$, $x_1$.

- Actually, one can turn the general equation to parametric equation in $R^2$ and conversely.
- General to parametric: Find two points in it and use the two-point vector equation.
  - $7x+5y=35$. (5,0) and (7,0).
  - $X=(1-t)(0,7)+t(5,0)$. $x=5t$, $y=7-7t$.
- Parametric to general: Eliminate t from the equation:
  - $x=5t$, $y= 7-7t$. Then $7x+5y = 35$. This is the general equation.
- Final comment: to give general equations for lines in 3-space, we need two equations.

# A plane in $R^3$

- From a plane S in $R^3$, we can obtain a point $x_0$ and a perpendicular vector n.
- From $x_0$, and n, we can obtain a *point-normal equation* of S:
  - $n.(x-x_0)=0.$
- Conversely, any x satisfying the equation lies in S.
- $(A,B,C).(x-x_0,y-y_0,z-z_0)=0.$
  - $A(x-x_0)+B(y-y_0)+C(z-z_0)=0.$
  - $Ax+By+Cz=D$. (general equation of S.)
  - Rmk:The coefficients give us the normal vector.

- Actually S passes 0 if and only if $D=0$.
- There is also a parametric equation of a plane:
  - Given a plane W, let $x_0$ be a point and let $v_1$ and $v_2$ be two vectors parallel to W.
  - Then $t_1v_1+t_2v_2$ is also parallel to W for any real numbers $t_1$ and $t_2$ by parallelogram laws.

    Thus $x_0+t_1v_1+t_2v_2$ lies in W.
  - Conversely, given any point x in W, $x- x_0$ is parallel to W and hence equals $t_1v_1+t_2v_2$ for some real numbers $t_1$ and $t_2$.
  - Thus $x=x_0+t_1v_1+t_2v_2$ is the equation of points of W.

- Examples: Given a point, and two vectors, find parametric equations.
- Given three points $x_0, x_1, x_2$ on W, find a parametric equation

  $$x = x_0 + t_1(x_1 - x_0) + t_2(x_2 - x_0).$$
- From general equation to a parametric equation. (Example 7)
  - Solution: is to find three distinct point and use the above.
- From parametric equation to a general equation.

(not yet studied.)

- In general $R^n$:
- A line through $x_0$ parallel to $v$:
  - $X=x_0+tv$.
- A plane through $x_0$ parallel to $v_1$, $v_2$.
  - $X=x_0+t_1v_1+t_2v_2$

- Actually, we can do s-dimensional subspace with s parallel vectors. But we stop here.
- See Example 8 (page 34)

# Comments on homework

- Ex set 1.2. Mostly computations.
- 1.2:13-16 use the definition
- 1.2: 32-35 Sigma notations (expect to know)
- 1.3: Two planes are parallel if their normal vectors are parallel. (perpendicular: the same)
- Finding normal vectors to the plane: Take the coefficients. (1.3:26-35)
- 1.3:37-38. Finding intersection line: Find two points in the intersections.
- 1.3:39-40. Use substitutions.