3.7. Matrix Factorization

LU decomposition
L: lower triangular
U: upper triangular

Solving a linear system by factorizations

- We try to write A=LU where L is lower triangular and U upper triangular.
- The reason is that the calculations are simpler.
  - LUx=b
  - Write Ux=y.
  - Ly=b and solve for y.
  - Solve for x in Ux=y
- Example 1.

Definition 3.7.1 A factorization of a square matrix A as $A = LU$, where L is lower triangular and U is upper triangular, is called an ***LU-decomposition*** or ***LU-factorization*** of A.

- If A can be reduced without using row exchanges, then we can obtain LU-decomposition.
  - $E_k \cdots . E_1 A = R$ ref R is upper triangular. Let U=R.
  - Thus $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$.
  - Then E_i^{-1} is either diagonal or is lower triangular.
  - The product $E_1^{-1} E_2^{-1} \ldots E_k^{-1}$ is lower triangular.

Theorem 3.7.2 *If a square matrix A can be reduced to row echelon form by Gaussian elimination with no row interchanges, then A has an LU-decomposition.*

- Steps to produce L and U.
  1. Reduce A to ref U without row changes while recording multipliers for leading 1s and multipliers to make 0s below the leading 1s.
  2. Diagonal of L: place the reciprocals of the multipliers of the leading 1s.
  3. Below the diagonals of L: place the negatives of multipliers to make 0.
  4. Use L and U.

- See Example 2.

The relation between Gaussian elimination and LU-decomposition

- Answer: They are equivalent for our matrices.
- Reason: As we do the row operations, LU-decomposition keeps track of operations.
- Gaussian elimination also keep track by changing b's.
- $Ax=b$ is changed to $Ux=y$. $Ly=b$ by multiplying L on both sides.
- That is $[A|b] \to [U|y]$.
- See Example 3. (omit)

Matrix inversion by LU-decompositions

- A nxn matrix
- AB=I can be converted to
- $A[x_1,\cdots,x_n] = [e_1,e_2,\cdots,e_n]$
- $Ax_1=e_1, Ax_2=e_2, \cdots, Ax_n=e_n$.
- We solve these by LU-decompositions.

LDU-decompositions

- We can write L=L'D where L' has only 1s in the diagonals.
- We can write A=LDU.
- See Example *.

Using permutation matrix.

- Sometimes, we can permute the rows of A so that LU-decomposition can happen.
- PA=U where P is a product of exchange elementary matrices.
- P is called a permutation matrix (it has only one 1 in each row or column)
- Acually P correspond to a 1-1 onto map f from {1,2,..,n} to itself. P_ij=1 if j=f(i) and 0 otherwise.

Computer cost to solve a linear system.

- Each operation +,−,/,* for floating numbers is a flop (floating point operation).
- We need to keep the number of flops down to minimize time.
- Today's PC : 10^9 flops per second.

- Solve $Ax=b$ by Gauss–Jordan method:
  - 1. n flops to introduce 1 in the first row
  - 2. n mult and n add to introduce one 0 below 1. There are $(n-1)$ rows: $2n(n-1)$ flops
  - Total for column 1 is $n+2n(n-1)=2n^2-n$.

- For next column, we replace n by n−1 and the total is $2(n-1)^2-(n-1)$.
- The forward total for columns: $2n^2+n+2(n-1)^2-(n-1)+\cdots+2-1$ $=2n^3/3+n^2/2-n/6$.
- Now backward stage:
- Last column (n−1) multiplication (n−1) addition to make 0 the entries above the leading 1s. Total: $2(n-1)$.
- For column (n−1): $2(n-2)$.
- Backward Total $2(n-1)+2(n-2)+\cdots+2(n-n)= n^2-n$.
- Total. $2n^3/3+3n^2/2-7n/6$.

For large examples

- Forward flops is approximately $2n^3/3$.
- Backward flops is approximately n^2.
- See Example 4 and Table 3.7.1.
- Actually choosing algorithms really depends on experiences for the particular set of problems.