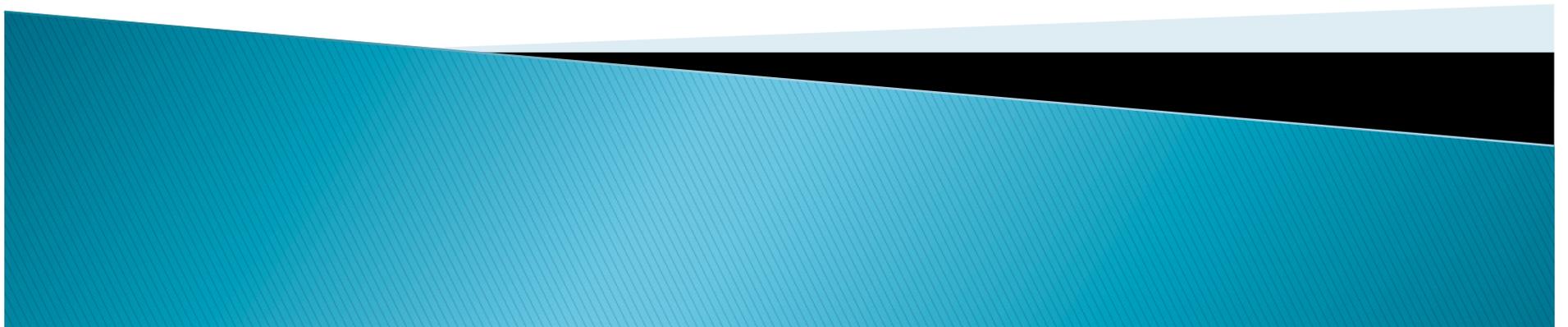


# Gauss–Seidel and Jacobi Iteration; Sparse linear system



# Iterative methods

- ▶ In many cases, such as finding eigenvalues, it is difficult to obtain an exact values.
- ▶ Alternatives is to use numerical methods.
- ▶ We iterate to find an approximate methods.
- ▶ Turn  $Ax = B$  to the equation  $x = Bx + c$ .
- ▶ To find the solution we start with some  $x_0$  almost arbitrarily chosen.
- ▶  $x_{(k+1)} = Bx_k + c$ .
- ▶ With right assumptions on  $A$ , this converges fast and we can estimate the error.



# Jacobi Iterations

- ▶  $Ax = b$ .
- ▶  $D$  the diagonal matrix made with diagonal entries of  $A$ . Assume these are nonzero.
- ▶ Thus  $D$  is invertible.
- ▶  $(A-D)x + Dx = b$
- ▶  $Dx = (D-A)x + b$
- ▶  $x = D^{-1}(D-A)x + D^{-1}b$
- ▶ So  $B = D^{-1}(D-A)$  and  $c = D^{-1}b$
- ▶  $x_{k+1} = D^{-1}(D-A)x_k + D^{-1}b$ .



- ▶ In terms of formula:
- ▶ Given  $Ax=b$ , we obtain as iteration equation:

$$x_1 = \frac{1}{a_{11}} (b_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}} (b_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n)$$

⋮

$$x_n = \frac{1}{a_{nn}} (b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{n(n-1)}x_{(n-1)})$$

- ▶ Example 1. (Large  $a_{ii}$ ,  $b_i$ , others small.)



# Gauss–Seidel Iterations

- ▶ While doing Gauss iterations, when new  $x_1$  is obtained, then use below.
- ▶ If new  $x_1, \dots, x_i$  is obtained, then use these below.
- ▶ This makes the convergence go faster.
- ▶ See Example 1.



# Gauss–Seidel matrix decomposition

- ▶  $A = D-L-U$  ( $L, U$  not from LU-decomposition but are the strict lower triangular part of  $A$  and the strict upper triangular part of  $A$ .)
- ▶  $D-L$  is invertible also.
- ▶ Then  $x_{k+1} = (D-L)^{-1}Ux_k + (D-L)^{-1}b$ .



# Convergence

- ▶ Strictly diagonally dominant if in each  $i$ -th row the absolute value of  $a_{ii}$  is larger than the sum of the absolute values of the other entries.

**Theorem 5.3.1** *If  $A$  is strictly diagonally dominant, then  $A\mathbf{x} = \mathbf{b}$  has a unique solution, and for any choice of the initial approximation the iterates in the Gauss–Seidel and Jacobi methods converge to that solution.*



# Speeding up convergence

- ▶ The size of dominance makes the convergence go faster.
- ▶ Today, some improvements exist: extrapolated Gauss–Seidel iterations.
- ▶ In general, Gauss iterations are really from Newtonian approximation methods....

