Gauss-Seidel and Jacobi Iteration; Sparse linear system

Iterative methods

- In many cases, such as finding eigenvalues, it is difficult to obtain an exact values.
- Alternatives is to use numerical methods.
- We iterate to find an approximate methods.
- Turn $Ax = B$ to the equation $x=Bx+c$.
- To find the solution we start with some x_0 almost arbitrarily chosen.
- $x_{(k+1)}=Bx_k + c$.
- With right assumptions on A, this converges fast and we can estimate the error.

Jacobi Iterations

- $Ax = b$.
- D the diagonal matrix made with diagonal entries of A. Assume these are nonzero.
- Thus D is invertible.
- $(A-D)x+Dx = b$
- $Dx = (D-A)x+b$
- $x=D^{-1}(D-A)x+D^{-1}b$
- So $B= D^{-1}(D-A)$ and $c=D^{-1}b$
- $x_k+1=D^{-1}(D-A)x_k + D^{-1}b$.

- In terms of formula:
- Given Ax=b, we obtain as iteration equation:

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{1n}x_n)$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n(n-1)}x_{(n-1)})$$

- Example 1. (Large a_ii, b_i, others small.)

Gauss-Seidel Iterations

- While doing Gauss iterations, when new x_1 is obtained, then use below.
- If new x_1,…,x_i is obtained, then use these below.
- This makes the convergence go faster.
- See Example 1.

Gauss-Seidel matrix decomposition

- $A = D - L - U$ (L,U not from LU-decomposition but are the strict lower triangular part of A and the strict upper triangular part of A.)
- $D-L$ is invertible also.
- Then $x_{k+1} = (D-L)^{-1}Ux_k + (D-L)^{-1}b$.

Convergence

- Strictly diagonally dominant if in each i-th row the absolute value of a_ii is larger than the sum of the absolute values of the other entries.

Theorem 5.3.1 *If A is strictly diagonally dominant, then $A\mathbf{x} = \mathbf{b}$ has a unique solution, and for any choice of the initial approximation the iterates in the Gauss–Seidel and Jacobi methods converge to that solution.*

Speeding up convergence

- The size of dominance makes the convergence go faster.
- Today, some improvements exist: extrapolated Gauss-Seidel iterations.
- In general, Gauss iterations are really from Newtonian approximation methods….