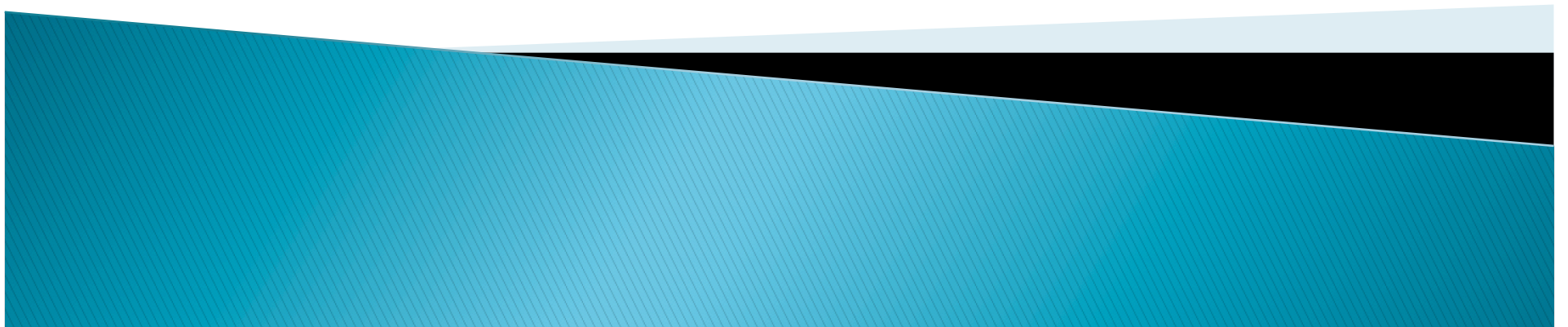


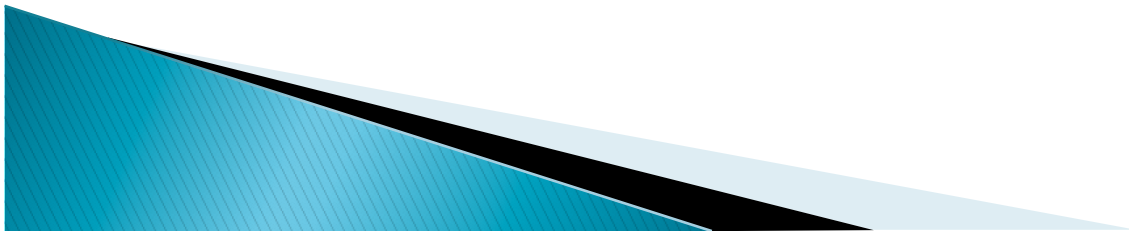
6.4. Composition and invertibility of linear transformations



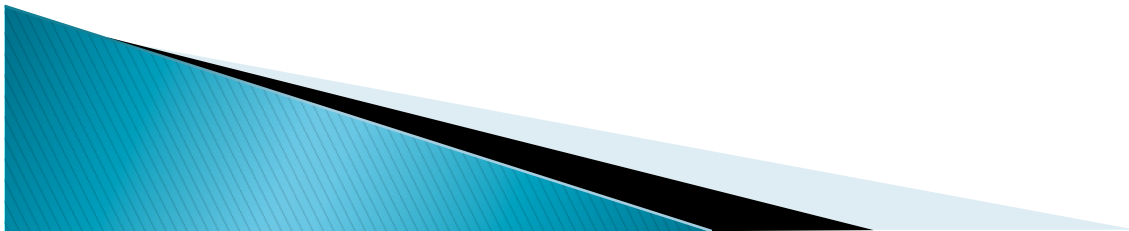
Compositions of linear transformations

- ▶ A composition of functions: $f: X \rightarrow Y, g: Y \rightarrow Z$, we obtain $g \circ f: X \rightarrow Z$.
- ▶ If f, g are linear, then $g \circ f$ is also linear.
- ▶ To verify, we need to show $+$ and scalar multiplications are preserved.

Theorem 6.4.1 *If $T_1: R^n \rightarrow R^k$ and $T_2: R^k \rightarrow R^m$ are both linear transformations, then $(T_2 \circ T_1): R^n \rightarrow R^m$ is also a linear transformation.*

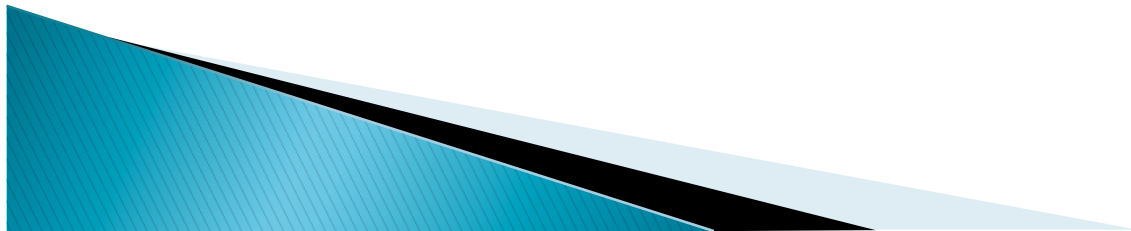


- ▶ Recall $T(x)=[T]x$ (p.272. (14))
- ▶ $(T_2 \cdot T_1)(e_i) = T_2(T_1(e_i)) = [T_2]([T_1](e_i)) = ([T_2][T_1])(e_i)$. (final step why?)
- ▶ Thus $[T_2 \cdot T_1] = [T_2][T_1]$. (why?)
- ▶ Conversely, given matrices A and B,
- ▶ $T_B \cdot T_A = T_{BA}$. (Let $T_2 = T_B, T_1 = T_A$).
- ▶ Example 1. $R_\theta \cdot R_\phi = R_{(\theta + \phi)}$. Verify using computations
- ▶ Example 2. $H_\theta \cdot H_\phi = R_2(\phi - \theta)$.
- ▶ Example 3. $T \cdot S$ may not equal $S \cdot T$. We can see that from matrices $T_A \cdot T_B = T_{AB}$. $T_B \cdot T_A = T_{BA}$. They would be equal iff $AB = BA$.

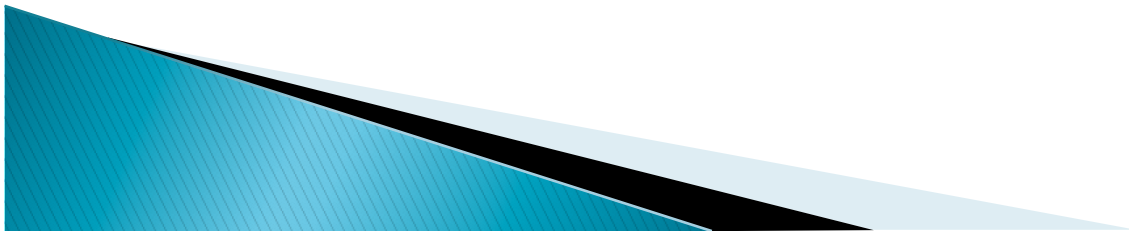


Compositions of three or more linear transformations.

- ▶ $T_1: \mathbb{R}^n \rightarrow \mathbb{R}^m, T_2: \mathbb{R}^m \rightarrow \mathbb{R}^l, T_3: \mathbb{R}^l \rightarrow \mathbb{R}^k$ We define $T_3 \cdot T_2 \cdot T_1: \mathbb{R}^n \rightarrow \mathbb{R}^k$ by
- ▶ $T_3 \cdot T_2 \cdot T_1(x) = T_3(T_2(T_1(x)))$.
- ▶ Since the compositions are associative, we have $(T_3 \cdot T_2) \cdot T_1 = T_3 \cdot (T_2 \cdot T_1)$. Thus we can drop the parantheses.
- ▶ $[T_3 \cdot T_2 \cdot T_1] = [T_3][T_2][T_1]$.
 - $[T_3 \cdot (T_2 \cdot T_1)] = [T_3][T_2 \cdot T_1] = [T_3]([T_2][T_1])$.
 - We use matrix multiplications are associative.
- ▶ $T_C \cdot T_B \cdot T_A = T_CBA$



- ▶ A classification:
 - A rotation in $\mathbb{R}^3 \leftrightarrow \det A = 1$.
 - A reflection composed with a rotation in $\mathbb{R}^3 \leftrightarrow \det A = -1$.
- ▶ A product of series of rotations is a rotation.
- ▶ A product of series of reflections and rotations with an even number of reflections is a rotation.
- ▶ A product of series of reflections and rotations with an odd number of reflections is a reflection composed with a rotation.



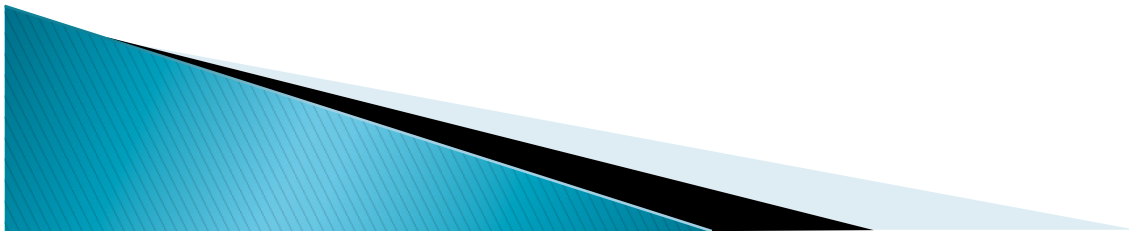
Yaw, pitch and roll

- ▶ Yaw: z-axis (up direction), pitch: x-axis (wing direction), roll: y-axis (the direction of travel)
- ▶ Corresponding rotations are $R_z\alpha, R_y\beta, R_x\gamma$.
- ▶ A composition of $R_z\alpha, R_y\beta, R_x\gamma$ can be achieved by a single rotation $R_v\delta$ in some direction of certain angle.
- ▶ Given these, we multiply them to get $R_v\delta$, and then find the axis direction v and the rotation δ (between 0 and π).
- ▶ See Example 5.
- ▶ Conversely, any rotation can be factored into yaw, pitch, roll rotations.



Factoring linear operators into compositions

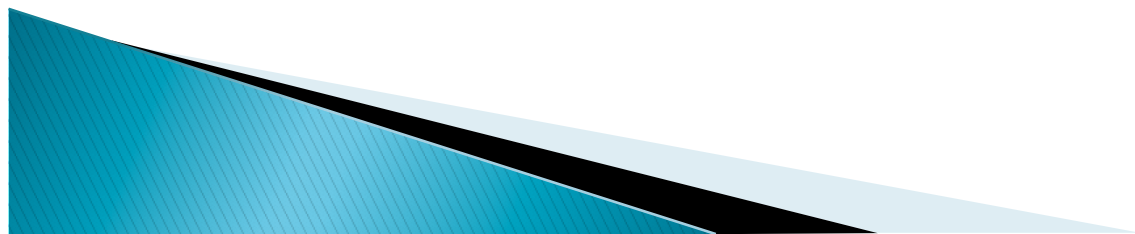
- ▶ We wish to factor a matrix into elementary pieces so that we can understand it better.
- ▶ For example, a diagonal operator can be understood as a composition of contraction and expansion along individual axis. E
- ▶ We restrict to \mathbb{R}^2 only.
- ▶ Example 7: There are five types of elementary matrices:



- (I) $[[1, k], [0, 1]]$ a shear in x -direction,
- (II) $[[1, 0], [k, 1]]$ a shear in y -direction,
- (III) $[[0, 1], [1, 0]]$ a reflection about $x=y$,
- (IV) $[[k, 0], [0, 1]]$ compression or expansion for $k \geq 0$.
- (V) $[[1, 0], [0, k]]$ same. For $k < 0$, they are compression or expansion followed by a reflection.

Theorem 6.4.4 *If A is an invertible 2×2 matrix, then the corresponding linear operator on R^2 is a composition of shears, compressions, and expansions in the directions of the coordinate axes, and reflections about the coordinate axes and about the line $y = x$.*

- ▶ Example 8: illustrates the factorization and how one can understand a linear transformation.



Inverse

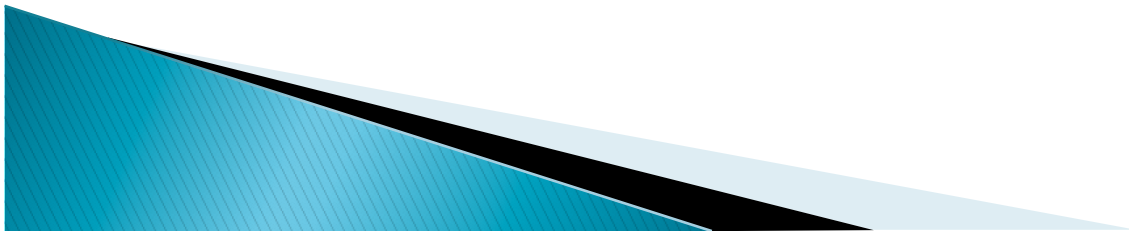
- ▶ $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$. Suppose it is one-to-one.
- ▶ Let w be in the range of T .
- ▶ Then there is a unique x in \mathbb{R}^n s.t. $T(x) = w$.
- ▶ Let $T^{-1}(w)$ be defined as x .
- ▶ $w = T(x) \iff x = T^{-1}(w)$ for w in $\text{range}(T)$.
- ▶ $T^{-1}: \text{range}(T) \rightarrow \mathbb{R}^n$.
- ▶ $TT^{-1} = \text{Id}$ on $\text{range}(T)$
- ▶ $T^{-1}T = \text{Id}$ on \mathbb{R}^n .

Theorem 6.4.5 *If T is a one-to-one linear transformation, then so is T^{-1} .*

Invertible linear operator

- ▶ If T is one-to-one and onto, then T^{-1} exists on the codomain, and is linear and one-to-one and onto. (The linearity already shown above. Other is just from the function theory)
- ▶ The matrix of T^{-1} is the inverse of the matrix of T .
 - $T^{-1}T(x) = [T^{-1}][T]x = x$. $[T^{-1}][T] = I$.

Theorem 6.4.6 *If T is a one-to-one linear operator on R^n , then the standard matrix for T is invertible and its inverse is the standard matrix for T^{-1} .*

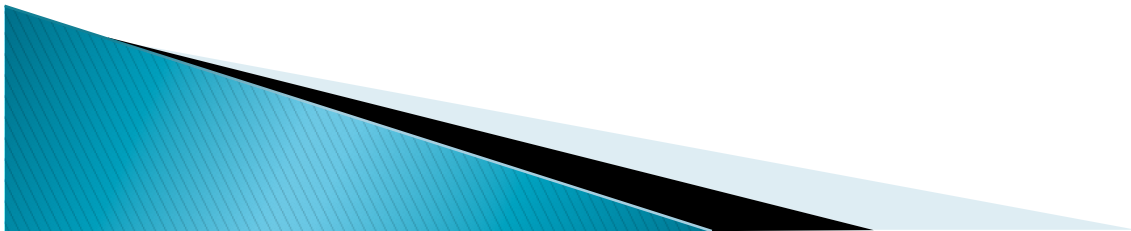


- ▶ $[T^{-1}] = [T]^{-1}$.
- ▶ $(T_A)^{-1} = T_{(A^{-1})}$.
- ▶ An inverse of a rotation in R^2 is a rotation with opposite angle.
- ▶ An inverse of a rotation in R^3 is a rotation with the same axis with an opposite angle or an opposite axis with the same angle.
- ▶ An inverse of an expansion by k in an axis direction is a contraction by $1/k$ in the same axis direction.
- ▶ An inverse of a reflection is the same reflection. $H_\theta H_\theta = I$.



Inverse and linear system

- ▶ $y = Ax$ given by a linear system as in (18).
- ▶ We have $x = A^{-1}y$ given by a linear system.
- ▶ We can obtain the second linear system by the first one by solving.
- ▶ Example 12.



Geometric properties of the invertible linear operators in \mathbb{R}^2 .

- ▶ What happens to lines, segments, polygons after acting by T ?

Theorem 6.4.7 *If $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an invertible linear operator, then:*

- (a) *The image of a line is a line.*
- (b) *The image of a line passes through the origin if and only if the original line passes through the origin.*
- (c) *The images of two lines are parallel if and only if the original lines are parallel.*
- (d) *The images of three points lie on a line if and only if the original points lie on a line.*
- (e) *The image of the line segment joining two points is the line segment joining the images of those points.*

Theorem 6.4.8 *If $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an invertible linear operator, then T maps the unit square into a nondegenerate parallelogram that has a vertex at the origin and has adjacent sides $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$. The area of this parallelogram is $|\det(A)|$, where $A = [T(\mathbf{e}_1) \ T(\mathbf{e}_2)]$ is the standard matrix for T .*