Dimension and structure

We will study subspaces through bases. Although there are many, they can be used to make a subspace like the Euclidean space.

This enables us to study abstract vector spaces later.

Bases for subspaces

- Consider V=Span{v_1,v_2,...,v_l}.
- If v_i is a linear combination of other vectors, we can drop v_i. V=Span{v_1,v_2,..,v_i-1,v_i+1,..,v_l}.
- To obtain a minimal set for a given V, we need to get V=Span{v_1,...,v_s} so that v_1,..,v_s are linearly independent.

Definition 7.1.1 A set of vectors in a subspace V of R^n is said to be a ***basis*** for V if it is linearly independent and spans V.

- Example: {O} no basis.
  - R^n itself is a subspace and has a standard basis.
  - A line through O has a basis consisting of only one unique vector. (One can choose any such.)
  - A plane through O has a basis consisting of two nonzero vectors tangent to the plane. Any two nonparallel and nonzero will form a basis.
  - To make the independence test easier, we use the following. That is we will only need to consider first i ones to understand independence.

Theorem 7.1.2 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of two or more nonzero vectors in R^n, then S is linearly dependent if and only if some vector in S is a linear combination of its predecessors.*

- The nonzero row vectors in the ref are linearly independent.

- Proof: Given a row with a leading 1 at jth position, the linear combinations of the previous rows, will give you a nonzero entry at entries below the j-th position. By theorem 7.1.2, we are done.

- Given an independent set of vectors {v_1,v_2,..,v_s}, suppose v is a nonzero vector which is not a linear combinations of the given ones, then one can add v to the list and the list is still independent. Why?

The existence of basis

Theorem 7.1.3 (***Existence of a Basis***) *If V is a nonzero subspace of R^n, then there exists a basis for V that has at most n vectors.*

- Proof: V is not {0}. Let v_1 be a nonzero vector. (It exists.)
  - If V=Span{v_1}, we are done.
  - If V is not Span{v_1}. Choose v_2 not in Span{v_1}. {v_1,v_2} are independent (why?). If V=Span{v_1,v_2}, then we are done.
  - Suppose we did this continuously, V has an independent set S={v_1,v_2,…,v_s}. If V=SpanS, then we are done. Otherwise, choose v_{s+1} not in the span.
  - By Theorem 3.4.8, s cannot be greater than n.
  - Thus we must stop at some s to get V=SpanS and S is independent.
- Basis is not unique for V.

Theorem 7.1.4 *All bases for a nonzero subspace of R^n have the same number of vectors.*

- Proof: {v_1,…,v_k}, {w_1,…,w_m} bases. Show k=m.
  - Assume k < m without loss of generality.
  - We can write w_i as linear combination of v_1,..,v_k.
  - Let A be kxm matrix doing this.
  - w_i = Σ_j A_jiv_j (*)
  - Then Ax=0 has a nontrivial solution since k < m.
  - Let (c_1,…,c_m) be the nontrivial solution.
  - Then c_1a_1+…+c_ma_m=0 for a_i ith row of A.
  - Then c_1w_1+…+c_mw_m =0 by computations.
  - This follows by pluging in (*) to the equation and collecting over v_is.

Dimension

Definition 7.1.5 If V is a nonzero subspace of R^n, then the ***dimension*** of V, written $\dim(V)$, is defined to be the number of vectors in a basis for V. In addition, we define the zero subspace to have dimension 0.

- Example: R^n has dimension n.
- Example: Solution space has dimension equal to the number f of free variables.
  - Setting i-th free variable 1 and the rest 0 gives us a column vector v_i. (canonical solutions)
  - Then {v_1,v_2,…,v_f} spans the solution space.
  - {v_1,v_2,..v_f} is linearly independent since the positions of 1 and 0 for free variable positions in v_is are different.
  - Thus {v_1,v_2,…,v_f} is a basis.
- See Example 7.

Dimension of a hyperplane

- a_1x_1+a_2x_2+…+a_nx_n=0.
- [1,*,*,…,*] or [0,1,*…,*],…
- It has n-1 free variables.

Theorem 7.1.6 *If* $\mathbf{a}$ *is a nonzero vector in* R^n, *then* $\dim(\mathbf{a}^{\perp}) = n - 1$.