7_4 The dimension theorem and its applications

Rank+nullity=dimension

The dimension theorem for matrices

- A be an mxn matrix.
- Ax=0. Let R be ref of A.
- r nonzero rows, n-r free variables.
- r is the rank of R and hence that of A.
- nullity A = nullity R = n-r
- rank A+ nullity A = n=the number of columns

Theorem 7.4.1 (***The Dimension Theorem for Matrices***) *If A is an $m \times n$ matrix, then*

$$\text{rank}(A) + \text{nullity}(A) = n \qquad (2)$$

- Example 1.

Expanding a linearly independent set to a basis

- {v_1,v_2,..,v_k} linearly independent in R^n.
- We can expend it to a basis.
  - First let A be the matrix with rows v_i.
  - The rank of A = k (Why?)
  - Solve Ax=0. The nullity A = n-k.
  - Find the basis of the solution space w_k+1, ..., w_n.
  - v_is and w_js are orthogonal.
  - {v_1,...,v_k,w_k+1,..,w_n} are linearly independent and hence is a basis.
  - Example 2: read yourself.

Some consequences

- This is a useful theorem. (See Example 3,4)

Theorem 7.4.2 *If an $m \times n$ matrix A has rank k, then:*

(a) *A has nullity $n - k$.*

(b) *Every row echelon form of A has k nonzero rows.*

(c) *Every row echelon form of A has $m - k$ zero rows.*

(d) *The homogeneous system $A\mathbf{x} = \mathbf{0}$ has k pivot variables (leading variables) and $n - k$ free variables.*

Theorem 7.4.3 (*The Dimension Theorem for Subspaces*) *If W is a subspace of R^n, then*

$$\dim(W) + \dim(W^{\perp}) = n \tag{3}$$

- Proof: If W={O}, trivially true. Suppose W is not {O}.
  - Form a matrix A with rows basis of W.
  - A is an mxn matrix. n is the dimension of R^n.
  - The row space of A is W.
  - The null space of A is W^c.
  - dim(W)+dim(W^c)=rank A + nullity A =n.

Theorem 7.4.4 *If A is an $n \times n$ matrix, and if T_A is the linear operator on R^n with standard matrix A, then the following statements are equivalent.*

(*a*) *The reduced row echelon form of A is I_n.*

(*b*) *A is expressible as a product of elementary matrices.*

(*c*) *A is invertible.*

(*d*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every vector* $\mathbf{b}$ *in* R^n.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every vector* $\mathbf{b}$ *in* R^n.

(*g*) $\det(A) \neq 0$.

(*h*) $\lambda = 0$ *is not an eigenvalue of A.*

(*i*) T_A *is one-to-one.*

(*j*) T_A *is onto.*

(*k*) *The column vectors of A are linearly independent.*

(*l*) *The row vectors of A are linearly independent.*

(*m*) *The column vectors of A span R^n.*

(*n*) *The row vectors of A span R^n.*

(*o*) *The column vectors of A form a basis for R^n.*

(*p*) *The row vectors of A form a basis for R^n.*

(*q*) $\text{rank}(A) = n$.

(*r*) $\text{nullity}(A) = 0$.

Hyperplanes

Theorem 7.4.5 *If W is a subspace of R^n with dimension $n-1$, then there is a nonzero vector $\mathbf{a}$ for which $W = \mathbf{a}^{\perp}$; that is, W is a hyperplane through the origin of R^n.*

Theorem 7.4.6 *The orthogonal complement of a hyperplane through the origin of R^n is a line through the origin of R^n, and the orthogonal complement of a line through the origin of R^n is a hyperplane through the origin of R^n. Specifically, if $\mathbf{a}$ is a nonzero vector in R^n, then the line $\text{span}\{\mathbf{a}\}$ and the hyperplane $\mathbf{a}^{\perp}$ are orthogonal complements of one another.*

Rank 1 matrices: classification

- If A is of rank 1, then nullity A=n-1.
  - The row space of A is a line through O.
  - The null space of A is a hyperplane.
  - The converse also holds.
- If rank A =1, then row space of A is spanned by a single vector a.
  - Each row vector is a scalar multiple of a.
  - The null space A is a^c.
  - The converse also holds.

- How to obtain a rank 1 matrix. One take a vector v and multiply by scalars u_1, u_2,..,u_m and obtain u_1v, u_2v,..,u_mv. Take A to be the mxn matrix with these rows.
- Then $A = uv^T$ for u=(u_1,u_2,...,u_m).
- Conversely, given a rank 1 matrix A, the rows of A are scalar multiple of some vector v. Listing the scalar multiples we form a vector u=(u_1,u_2,...,u_m).
- We obtain $A= uv^T$ (See Example 8)

Theorem 7.4.7 *If* $\mathbf{u}$ *is a nonzero* $m \times 1$ *matrix and* $\mathbf{v}$ *is a nonzero* $n \times 1$ *matrix, then the outer product*

$$A = \mathbf{u}\mathbf{v}^T$$

has rank 1. *Conversely, if* A *is an* $m \times n$ *matrix with rank* 1, *then* A *can be factored into a product of the above form.*

Symmetric rank 1 matrices

- $A=uu^T$ is symmetric. ($A^T=uu^T$ also.)

Theorem 7.4.8 *If* $\mathbf{u}$ *is a nonzero* $n \times 1$ *column vector, then the outer product* $\mathbf{u}\mathbf{u}^T$ *is a symmetric matrix of rank* 1*. Conversely, if* A *is a symmetric* $n \times n$ *matrix of rank* 1*, then it can be factored as* $A = \mathbf{u}\mathbf{u}^T$ *or else as* $A = -\mathbf{u}\mathbf{u}^T$ *for some nonzero* $n \times 1$ *column vector* $\mathbf{u}$.