# 7.6 The pivot theorem

# Basis problem

- Now address the problem of extracting a basis in S for the Span(S).
- The row operations changes the column spaces.
- If A and B are row equivalent, then $Ax=0$, $Bx=0$ have the same set of solutions.
- $Ax=0 <-> x_1a_1+x_2a_2+\ldots+x_na_n=0$.
- $Bx=0 <-> x_1b_1+x_2b_2+\ldots+x_nb_n=0$.

**Theorem 7.6.1** *Let A and B be row equivalent matrices.*

(*a*) *If some subset of column vectors from A is linearly independent, then the corresponding column vectors from B are linearly independent, and conversely.*

(*b*) *If some subset of column vectors from B is linearly dependent, then the corresponding column vectors from A are linearly dependent, and conversely. Moreover, the column vectors in the two matrices have the same dependency relationships.*

- Proof: If necessary form A’ from the set of column vectors of A.
- Thus our strategy is to ref A and choose the pivot columns as basis and transfer back to A.
- Example 1.

# Pivot theorem

**Definition 7.6.2** The column vectors of a matrix $A$ that lie in the column positions where the leading 1's occur in the row echelon forms of $A$ are called the ***pivot columns*** of $A$.

**Theorem 7.6.3** (***The Pivot Theorem***) *The pivot columns of a nonzero matrix $A$ form a basis for the column space of $A$.*

- Proof: We see that leading 1s are at every position in the pivot column vectors.

# Pivot algorithm

**Algorithm 1** If $W$ is the subspace of $R^n$ spanned by $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$, then the following procedure extracts a basis for $W$ from $S$ and expresses the vectors of $S$ that are not in the basis as linear combinations of the basis vectors.

**Step 1.** Form the matrix $A$ that has $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ as successive column vectors.

**Step 2.** Reduce $A$ to a row echelon form $U$, and identify the columns with the leading 1's to determine the pivot columns of $A$.

**Step 3.** Extract the pivot columns of $A$ to obtain a basis for $W$. If appropriate, rewrite these basis vectors in comma-delimited form.

**Step 4.** If it is desired to express the vectors of $S$ that are not in the basis as linear combinations of the basis vectors, then continue reducing $U$ to obtain the reduced row echelon form $R$ of $A$.

**Step 5.** By inspection, express each column vector of $R$ that does not contain a leading 1 as a linear combination of preceding column vectors that contain leading 1's. Replace the column vectors in these linear combinations by the corresponding column vectors of $A$ to obtain equations that express the column vectors of $A$ that are not in the basis as linear combinations of basis vectors.

# Example 2

- Given W=span(S). S finite.
- (a) Extract basis in S.
- (b) Express other vectors in S

# Basis for the fundamental spaces

- A mxn -> U upper echelon -> R ref.
- 1. row(A): basis nonzero rows of U or R.
- 2. col(A): pivot columns of A.
- 3. null(A): canonical solutions from Rx=0.
- 4. null($A^T$): Solve $A^T x=0$.
- A mxn rank k. dim null($A^T$)= m-k. Why? If k=m, dim=0.
- Another method using row operations only.

**Algorithm 2** If $A$ is an $m \times n$ matrix with rank $k$, and if $k < m$, then the following procedure produces a basis for null$(A^T)$ by elementary row operations on $A$.

**Step 1.** Adjoin the $m \times m$ identity matrix $I_m$ to the right side of $A$ to create the partitioned matrix $[A \mid I_m]$.

**Step 2.** Apply elementary row operations to $[A \mid I_m]$ until $A$ is reduced to a row echelon form $U$, and let the resulting partitioned matrix be $[U \mid E]$.

**Step 3.** Repartition $[U \mid E]$ by adding a horizontal rule to split off the zero rows of $U$. This yields a matrix of the form

$$\underset{\quad n \qquad\quad m}{\left[\begin{array}{c|c} V & E_1 \\ \hline 0 & E_2 \end{array}\right]}\begin{array}{l} k \\ m-k \end{array}$$

where the margin entries indicate sizes.

**Step 4.** The row vectors of $E_2$ form a basis for null$(A^T)$.

- Example 3:

# Column-row factorization

**Theorem 7.6.4** (***Column-Row Factorization***) *If $A$ is a nonzero $m \times n$ matrix of rank $k$, then $A$ can be factored as*

$$A = CR \tag{1}$$

*where $C$ is the $m \times k$ matrix whose column vectors are the pivot columns of $A$ and $R$ is the $k \times n$ matrix whose row vectors are the nonzero rows in the reduced row echelon form of $A$.*

- Proof: EA=R_0. E mxm matrix a product of elementary matrices.
  - R_0 ref of A. mxn-matrix
  - Let R be the kxn-matrix of nonzero rows of R.
  - Then let $E^{-1}$=[C|D] C mxk. D mx(m-k)

$$R_0 = \left[\frac{R}{O}\right]$$

- Proof continued:
- A=E$^{-1}$R=

$$[C \mid D]\left[\frac{R}{O}\right] = CR + DO = CR$$

- C consists of pivot columns of A.
  - Multiplying by E$^{-1}$ to R_0 returns to A.
  - Restrict to pivot columns of R -> pivot columns of A.
  - Pivot columns of R form I of kxk size.
  - CR restricted CI=C. Thus C is the pivot columns of A.
- Example 4.

# Column-row expansion

- We can write the above as the sum of vector products…

**Theorem 7.6.5** (***Column-Row Expansion***) *If $A$ is a nonzero matrix of rank $k$, then $A$ can be expressed as*

$$A = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_k\mathbf{r}_k \tag{4}$$

*where $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k$ are the successive pivot columns of $A$ and $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k$ are the successive nonzero row vectors in the reduced row echelon form of $A$.*